Polynomial Time SAT Decision, Hypergraph Transversals and the Hermitian Rank

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Abstract. Combining graph theory and linear algebra, we study SAT problems of low "linear algebra complexity", considering formulas with bounded hermitian rank. We show polynomial time SAT decision of the class of formulas with hermitian rank at most one, applying methods from hypergraph transversal theory. Applications to heuristics for SAT algorithms and to the structure of minimally unsatisfiable clause-sets are discussed.

1 Introduction

Connections between graphs, clause-sets and matrices based on conflicting literals have been investigated in [13], introducing the notions "hermitian rank" and "hermitian defect" into the field of SAT-related problems. In this article we continue these investigations, and we present new SAT decision algorithms combining graph theory and linear algebra.

The *conflict multigraph* $\mathrm{cmg}(F)$ of a clause-set F ([14, 13]) has the clauses of F as vertices, and as many (parallel) edges joining two vertices as the clauses have conflicts, i.e., clashing literals. The *hermitian rank* $h(F)$, as adopted in [14, 13] from [9], is the hermitian rank of the adjacency matrix of $\mathrm{cmg}(F)$, where the hermitian rank $h(A)$ of a symmetric real matrix A is the maximum of the number of positive and negative eigenvalues of A, and can also be naturally computed from the sign changes in the characteristic polynomial of A ([15]). The *hermitian defect* $m - h(A)$, where m is the dimension of A, equals the Witt index of the quadratic form associated with A ([15]). We explore the use of the conflict multigraph $\mathrm{cmg}(F)$ of a clause-set F for SAT algorithms. On the one hand, we exploit structural properties of $\mathrm{cmg}(F)$, namely bipartiteness and variations, and on the other hand we use the hermitian rank $h(F)$ as a complexity measure for SAT decision.

1.1 Using the hermitian rank as complexity measure

We investigate, whether $h(F)$ can be used as a measure of problem complexity for SAT decision, yielding a SAT decision algorithm whose running time depends mainly and monotonically on $h(F)$. The hermitian rank is at most the number of variables, i.e., $h(F) \leq n(F)$, as shown in [14, 13], reformulating the original Graham-Pollak theorem [8]. It would be interesting to prove an upper bound $2^{h(F)}$ on time complexity of SAT decision (ignoring polynomial factors). Given that $h(F)$ can be computed in polynomial time using the symmetric form of Gaussian elimination ([2]), the upper bound $2^{h(F)}$ would follow using the framework proposed in [11, 17, 18]: Use $h(F)$ as the

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heuristic of a DLL-like SAT algorithm, i.e., chose a branching variable where in both branches $h(F)$ strictly decreases (as much as possible). In Section 5 we further discuss this approach.

In this article we concentrate on the case of SAT decision for clause-sets F with $h(F) \leq 1$, which, besides the trivial case $h(F) = 0$ equivalent to $n(F) = 0$, constitutes the base case of the above approach. Let $\mathcal{CLS}_h(1)$ denote the set of clause-sets F with $h(F) \leq 1$. Since the hermitian rank is computable in polynomial time, $\mathcal{CLS}_h(1)$ is decidable in polynomial time. In [14, 13] it was shown that application of partial assignments does not increase the hermitian rank (using Cauchy's interlacing inequalities), i.e., $h(\varphi * F) \leq h(F)$ holds for any clause-set F and partial assignment φ. In the special case of $h(F) = 1$ we either have $n(\varphi * F) = 0$ (i.e., $\varphi * F$ is trivial) or $h(\varphi * F) = 1$, and we can not use SAT decision based on splitting, but a different approach is needed.

The hermitian rank $h(A)$ of a symmetric real matrix is the minimal rank of some matrix B with $A = B + B^t$ ([14, 13]). This property is basic to infer in Lemma 2 a characterisation of symmetric matrices with zero diagonal and hermitian rank one, which in turn yields **Theorem 3**, which states, that every $F \in \mathcal{CLS}_h(1)$ after elimination of blocked clauses ([12, 11]) has as conflict multigraph a complete bipartite *graph* (having no parallel edges). This special structure is exploited for SAT decision of F as explained in the following subsection.

Following the program of characterising the (hereditary) classes of graphs with at most k positive respectively negative eigenvalues via forbidden induced subgraphs, in [19, 20] it is shown that a connected *graph* G (i.e., its adjacency matrix A) has at most one negative eigenvalue (that is, $h(A) \leq 1$) if and only if it is a complete bipartite graph (see Theorem 1.34 in [4]). Our characterisation in Theorem 3 differs by its approach based on linear algebra, and by considering the general case of *multi*graphs, where there is no characterisation in terms of a finite number of forbidden induced subgraphs, and where the elimination of blocked clauses becomes essential.

1.2 Using the transversal hypergraph problem for SAT decision

In **Theorem 3** we characterised clause-sets with hermitian rank one as having a conflict multigraph which is a complete bipartite graph after elimination of blocked clauses. We will use this characterisation for fast SAT decision.

In Section 4 we define *bipartite clause-sets* as clause-sets with a bipartite conflict multigraph, and we characterise them using basic results from algebraic graph theory. The main case of bipartite clause-sets is given by *positive-negative clause-sets* (PN-clause-sets) where every clause either is positive or negative. The SAT problem for the class of PN-clause-sets is easily seen to be NP-complete. Therefore we need to refine the class of bipartite clause-sets to obtain a class with feasible SAT decision capturing clause-sets of hermitian rank one.

A *bi-hitting clause-set* has a conflict multigraph which is a complete bipartite multigraph (that is, every pair of vertices from different parts is connected by at least one edge). Bi-hitting PN-clause-sets constitute again the core part of the class of bi-hitting clause-sets. From a bi-hitting PN-clause-set F we extract two hypergraphs $\mathcal{H}_\mathrm{P}$ resp. $\mathcal{H}_\mathrm{N}$ by considering the set of positive clauses resp. the set of negative clauses. If F does not contain subsumed clauses, then F is unsatisfiable if and only if $(\mathcal{H}_\mathrm{P}, \mathcal{H}_\mathrm{N})$ is a *transversal hypergraph pair*, that is, $\mathcal{H}_\mathrm{N}$ is the set of (minimal) transversals of $\mathcal{H}_\mathrm{P}$ (and vice versa). Whether the transversal hypergraph problem is solvable in polynomial time is an important open problem ([6]). Recently it was shown to be solvable in quasi-polynomial time ([7]), and thus the SAT problem for bi-hitting clause-sets is solvable in quasi-polynomial time.

The final step exploits, that Theorem 3 not only yields complete bipartite multigraphs, but complete bipartite *graphs* (no parallel edges). Clause-sets where the conflict multigraph is a complete bipartite graph are called 1-*uniform bi-hitting clause-sets*, and in Lemma 11 we show that the SAT problem for this class is essentially the same as the *exact transversal hypergraph problem*. This problem was investigated in [5] and shown to be solvable in polynomial time; hence the SAT problem for clause-sets with hermitian rank one is decidable in polynomial time (**Theorem 13**).

The paper is organised as follows: We start with the preliminaries in Section 2. In Section 3 we characterise the class of clause-sets with hermitian rank one. In Section 4 the relations between (uniform) bi-hitting clause-sets and the (exact) hypergraph transversal problem is examined. Finally open problems and directions for future research are discussed in Section 5.

2 Preliminaries

We assume a universe $\mathcal{VA}$ of variables, from which literals (negated and unnegated variables) are constructed, using $\overline{x}$ for the negation of a literal x. A clause-set is a finite sets of clauses, where a clause is a finite and complement-free set of literals, denoting the set of all clause-sets by $\mathcal{CLS}$. For $F \in \mathcal{CLS}$ the number of variables is $n(F)$ and the number of clauses is $c(F)$. A partial assignment is a map $\varphi : V \to \{0,1\}$ for some $V \subseteq \mathcal{VA}$, and the application of φ to F is denoted by $\varphi * F \in \mathcal{CLS}$. Given some class $\mathcal{C} \subseteq \mathcal{CLS}$ of clause-sets and some "measure" $f : \mathcal{C} \to \mathbb{R}$, by $\mathcal{C}_f(b) := \{F \in \mathcal{C} : f(F) \le b\}$ we denote the set of clause-sets in $\mathcal{C}$ with measure at most $b \in \mathbb{R}$.

A hypergraph is a pair $(V, \mathbb{H})$, where V is the set of vertices and $\mathbb{H}$ is a set of subsets of V (the "hyperedges"). A transversal T of a hypergraph $(V, \mathbb{H})$ is a subset $T \subseteq V$ with $T \cap H \neq \emptyset$ for all $H \in \mathbb{H}$, while a minimal transversal is a transversal such that no strict subset is also a transversal. The set of all minimal transversals of $(V, \mathbb{H})$ is $\mathrm{Tr}(V, \mathbb{H})$, and fulfils $\mathrm{Tr}(\mathrm{Tr}(\mathbb{H})) = \mathbb{H}$ if $\mathbb{H}$ is simple, that is, does not contain subsumed hyperedges. An independent set of $(V, \mathbb{H})$ is a subset $I \subseteq V$ such that there is no hyperedge $H \in \mathbb{H}$ with $H \subseteq I$. For more information on hypergraphs see for example [1]. A permutation matrix is a square matrix over $\{0,1\}$ such that every row and every column contains exactly one entry equal to 1. Transposition of matrices A is denoted by A^t.

The *conflict multigraph* of $F \in \mathcal{CLS}$ is denoted by $\mathrm{cmg}(F)$; the vertices of $\mathrm{cmg}(F)$ are the clauses of F, and clauses $C, D \in F$ are joined by exactly $|C \cap \overline{D}|$ parallel edges, using $\overline{D} = \{\overline{x} : x \in D\}$. The *symmetric conflict matrix* $\mathrm{C_s}(F)$ is the adjacency matrix of $\mathrm{cmg}(F)$. The *hermitian rank* $h(F)$ has many equivalent characterisations, for example $h(F) = \max(i_-(\mathrm{C_s}(F)), i_+(\mathrm{C_s}(F)))$, where $i_\pm(A)$ for a real square matrix A denotes the number of positive resp. negative eigenvalues (see [14, 13, 15] for further information).

In [14, 13] multi-clause-sets have been have been considered instead of clause-sets, since for example for applications of matching theory it is important to avoid the contraction of multiple clauses, and furthermore for example the symmetric conflict number $\mathrm{n_s}(A)$ of a matrix A is defined as the minimal $n(F)$ for *multi*-clause-sets with $\mathrm{C_s}(F) = A$, since possible repetition of clauses is needed here to apply algebraic methods. However, in our context there is no need for using multi-clause-sets (and no advantage), and thus in this paper only clause-sets are considered. One last comment on a potential difference between clause-sets and multi-clause-sets: Consider a clause-set $F \in \mathcal{CLS}$ and a partial assignment φ, and let us denote by $F' \in \mathcal{MCLS}$ the corresponding multi-clause-set. Now $\varphi * F \in \mathcal{CLS}$ is obtained from $\varphi * F' \in \mathcal{MCLS}$ by contracting multiple clauses (in the computation of $\varphi * F'$ no contraction takes place), which could make a difference for certain measures. However it can easily be seen that contraction of multiple clauses does not affect $i_\pm$, and thus can be applied freely in our context.

3 Characterisation of clause-sets with hermitian rank one

Since $h(F)$ can be computed in polynomial time, polynomial time decision of the class $\mathcal{CLS}_h(1)$ follows. The characterisation of matrices with hermitian rank one uses the following basic lemma on "combinatorial linear algebra".

Lemma 1. *Consider a real square matrix A with zeros on the diagonal and with* $\mathrm{rank}(A) = 1$*. Then there is a permutation matrix P and a matrix B with* $\mathrm{rank}(B) = 1$ *having no zero entry such that* $P^t \cdot A \cdot P = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$.

Proof. The elementary matrix operation we use is row resp. column exchange followed by the corresponding column resp. row exchange, i.e., we exchange rows i and j, immediately followed by exchange of columns i and j, or vice versa — in both ways we get the same result. We speak of a *combined row/column exchange* resp. a *combined column/row exchange*. Eventually, when we transformed A into $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, multiplying all matrices together representing the column exchanges we obtain P, and multiplying all matrices together representing the row exchanges we obtain P^t. Note all matrices obtained by applying combined row/column exchanges from A have zero diagonal.

Let the order of A be $m \geq 1$. Our aim is to move all non-zero entries of A to the upper right corner. There exists a non-zero row in A, and by a combined row/column exchange we get A' having a non-zero first row. Now we apply combined column/row exchanges for columns i, j with $A'_{1,i} \neq 0$, $A'_{1,j} = 0$ and $i < j$ until the first row is ordered in such a way that first come all zero entries and then all non-zero entries (note that due to $A_{1,1} = 0$ we always have $i, j \geq 2$ in this process, and thus the first row as a whole stays untouched). We obtain a matrix A'' having a column index $2 \leq k^* \leq m$ with the property that $A''_{1,i} = 0$ for all $1 \leq i \leq k^*$ and $A''_{1,i} \neq 0$ for all $k^* \leq i \leq m$. For the purpose of this proof, we call a matrix A^* with this property a k^*-matrix. Any k^*-matrix A of rank one has the following properties:

Every row of A is a multiple of the first row, and thus we have $A_{i,j} = 0$ for all indices i, j with $j < k^*$. If some row i contains a non-zero entry $A_{i,j} \neq 0$ (thus $j \geq k^*$), then actually for all $k^* \leq j' \leq m$ we have $A_{i,j'} \neq 0$, which can be seen as follows: There is $\lambda \in \mathbb{R}$ with $\lambda \cdot A_{1,*} = A_{i,*}$. If there would be some $k^* \leq j' \leq m$ with $A_{i,j'} = 0$, then due to $\lambda \cdot A_{1,j'} = A_{i,j'}$ and $A_{1,j'} \neq 0$ we would have $\lambda = 0$ contradicting $\lambda \cdot A_{1,j} = A_{i,j} \neq 0$.

Now consider a zero row $A''_{i,*}$ and some non-zero row $A''_{i',*}$ with $i < i'$. Since $A''_{i',i'} = 0$ and A'' has the k^*-property, $i' < k^*$ must hold. Performing the combined row/column exchange on A'' for rows i and i' maintains the k^*-property. Repeating this process until every zero row is below any non-zero row we obtain a matrix of the form $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ with all entries of B non-zero. □

As already mentioned in the introduction, the hermitian rank $h(A)$ of a symmetric real matrix is the minimal rank of some matrix B with $A = B + B^t$ ([14, 13]).

Lemma 2. *Consider a symmetric real matrix A with zero diagonal. Then $h(A) \leq 1$ iff there is a real matrix B with* $\mathrm{rank}(B) \leq 1$ *and a permutation matrix P with $A = P^t \cdot \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \cdot P$.*

Proof. First assume $h(A) \leq 1$. If $h(A) = 0$, then $A = 0$, and thus we can choose $B := 0$ and $P := I$. So assume $h(A) = 1$. Thus there exists a matrix B_0 with $\mathrm{rank}(B_0) = 1$ and $B_0 + B_0^t = A$. It has B_0 a zero diagonal, and thus by Lemma 1 there exists B with $\mathrm{rank}(B) = 1$ and only non-zero entries, and a permutation matrix P such that $P^t \cdot B_0 \cdot P = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. Since also $P^t \cdot B_0 \cdot P$ has a zero diagonal, it is B located in $P^t \cdot B_0 \cdot P$ above the diagonal, and thus B^t in $(P^t \cdot B_0 \cdot P)^t = \begin{pmatrix} 0 & 0 \\ B^t & 0 \end{pmatrix}$ is located below the diagonal:

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B^t & 0 & 0 \end{pmatrix}.$$

Using $P^{-1} = P^t$ we have

$$P \cdot \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B^t & 0 & 0 \end{pmatrix} \cdot P^t = P \cdot \left(\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^t \right) \cdot P^t = P \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot P^t + P \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^t \cdot P^t = B_0 + B_0^t = A.$$

On the other hand, if $A = P^t \cdot \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \cdot P$ for some B with $\mathrm{rank}(B) \leq 1$ and some permutation matrix P, then with $B_0 := P^t \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot P$ obviously $\mathrm{rank}(B_0) = \mathrm{rank}(B) \leq 1$ and $B_0 + B_0^t = A$, thus $h(A) \leq \mathrm{rank}(B_0) \leq 1$. □

We remind at the notion of a blocked clause w.r.t. a clause-set F ([11, 12]), which is a clause C containing a literal $x \in C$ such that for all clauses $D \in F$ with $\overline{x} \in D$ there exists a literal $y \in C \setminus \{x\}$ with $\overline{y} \in D$. Blocked clauses can be eliminated satisfiability-equivalently. If a clause-set F not containing the empty clause does not contain a blocked clause, then obviously every row and every column of $\mathrm{C_s}(F)$ contains at least one entry equal to one. An easy fact from graph theory is, that a multigraph G is a complete bipartite graph iff there exists a permutation matrix P and a matrix B with all entries equal to 1 such that the adjacency matrix of G is $P^t \cdot \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \cdot P$.

Theorem 3. *Consider $F \in \mathcal{CLS}_h(1)$ with $\bot \notin F$, and obtain $F' \in \mathcal{CLS}_h(1)$ from F by elimination of (all) blocked clauses. Then the conflict multigraph* $\mathrm{cmg}(F')$ *is a complete bipartite graph (without parallel edges).*

Proof. We have $F' \in \mathcal{CLS}_h(1)$, since elimination of clauses can not increase the hermitian rank ([14, 13]). If $F' = \top$, then the assertion is trivial. So we assume that F' contains at least one clause. Since F' does not contain a blocked clause, every row and every column of $\mathrm{C_s}(F')$ contains at least one entry equal to 1. Applying Lemma 2 to $C_s(F')$ we obtain $C_s(F') = P^t \cdot \left(\begin{smallmatrix} 0 & B \\ B^t & 0 \end{smallmatrix}\right) \cdot P$ for some permutation matrix P and a matrix B with $\mathrm{rank}(B) = 1$. There are no zero rows or columns in B, and since the rank of B is one, every row (resp. column) of B is a non-zero multiple of every other row (resp. column). We want to show that every entry of B is 1. So assume that there are indices i, j with $B_{i,j} \neq 1$. We have $B_{i,j} \neq 0$, since otherwise every entry in row i would be zero, using that every column j' of B is a multiple of column j. Thus $B_{i,j} \geq 2$. We know that there is a column index j' with $B_{i,j'} = 1$ and a row index i' with $B_{i',j} = 1$. Thus column j multiplied with $1/B_{i,j}$ yields column j', and thus $B_{i',j}/B_{i,j} = B_{i',j'}$, but $0 < B_{i',j}/B_{i,j} = 1/B_{i,j} < 1$ contradicting integrality of $B_{i',j'}$. □

4 SAT decision, conflict multigraphs and the transversal hypergraph problem

In this section we investigate the problem, how the fact that the conflict multigraph of a clause-set is a complete bipartite graph can be exploited for efficient SAT decision. We proceed in three stages: First we consider clause-sets with bipartite conflict multigraphs in general, then the case of *complete* bipartite conflict multigraphs is investigated, and finally we turn to the case where the conflict multigraph is a complete bipartite *graph*.

4.1 Bipartite clause-sets and PN-pairs

$F \in \mathcal{CLS}$ is called **bipartite** if the conflict multigraph of F is bipartite. This is easily seen to be equivalent to the existence of a permutation matrix P and a matrix B such that $C_s(F) = P^t \cdot \left(\begin{smallmatrix} 0 & B \\ B^t & 0 \end{smallmatrix}\right) \cdot P$ holds. Immediately from a well-known characterisation of bipartite (multi-)graphs in algebraic graph theory (see for example Theorem 3.11 in [4] together with the footnote) we get

Lemma 4. *A clause-set $F \in \mathcal{CLS}$ is bipartite if and only if the following two conditions hold:*

1. $i_-(F) = i_+(F) = h(F)$*;*
2. *for* $1 \leq i \leq h(F)$ *we have* $\theta_i(F) = \theta_{c(F)-i+1}(F)$*.*

Since the sum of the eigenvalues of a symmetric real matrix is equal to the trace of the matrix (the sum of the diagonal elements), and (symmetric) conflict matrices have a zero diagonal, it is easy to see that a clause-set F fulfils $h(F) = 1$ iff $i_+(F) = i_-(F) = 1$ holds and the absolute values of the positive and the negative eigenvalue coincide. Thus Lemma 4 yields an alternative proof that clause-sets in $\mathcal{CLS}_h(1)$ are bipartite as proven directly in Theorem 3. Actually Theorem 3 proved much more, and we will now see that the property of a clause-set being bipartite alone does not help much for satisfiability decision.

A clause is called *positive* resp. *negative* if it only contains positive resp. negative literals. A clause-set $F \in \mathcal{CLS}$ is called *positive-negative* ("PN-clause-set" for short) if for every $C \in F$ we have, that C is positive or negative. Obviously, every positive-negative clause-set is bipartite. The Pigeonhole formulas are examples of positive-negative clause-sets. By introducing new variables, every clause-set can be transformed in linear time into a satisfiability-equivalent positive-negative clause-set, and thus satisfiability decision for bipartite clause-sets is NP-complete.

Intuitively, the class of PN-clause-sets is the core of the class of bipartite clause-sets. We make this more precise to clarify the relationship to the hypergraph transversal problem. A *PN-pair* is a pair $(\mathbb{H}_1, \mathbb{H}_2)$, where each $\mathbb{H}_i \subseteq \mathbb{P}(\mathcal{VA})$ is a set of hyperedges considered as a hypergraph with vertex set the set of variables appearing in it. For a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ we define $F(\mathbb{H}_1, \mathbb{H}_2) := \mathbb{H}_1 \cup \{\overline{H} : H \in \mathbb{H}_2\}$, that is, the positive clauses of the PN-clause-set $F(\mathbb{H}_1, \mathbb{H}_2)$ are given by the elements of $\mathbb{H}_1$, while the negative clauses are given by the elements of $\mathbb{H}_2$ with elementwise complemented literals. For example $F(\{\{a, b\}, \{a, c\}\}, \{\{b, c\}\}) = \{\{a, b\}, \{a, c\}, \{\overline{b}, \overline{c}\}\}$. As noticed in [6]:

Lemma 5. *Consider a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$. Then $F(\mathbb{H}_1, \mathbb{H}_2)$ is unsatisfiable if and only if for all $H_1 \in \mathrm{Tr}(\mathbb{H}_1)$ there exists $H_2 \in \mathbb{H}_2$ with $H_2 \subseteq H_1$, or equivalently, $F(\mathbb{H}_1, \mathbb{H}_2)$ is satisfiable iff there exists $H_1 \in \mathrm{Tr}(\mathbb{H}_1)$ such that H_1 is an independent set of $\mathbb{H}_2$.*

Proof. If $F := F(\mathbb{H}_1, \mathbb{H}_2)$ is satisfiable, then there is a partial assignment φ with $\varphi * F = \top$. Let $V_1 := \{v \in \mathrm{var}(\varphi) : \varphi(v) = 1\}$. It is V_1 a transversal of $\mathbb{H}_1$, and there is no $H_2 \in \mathbb{H}_2$ with $H_2 \subseteq V_1$, i.e., V_1 is an independent set of $\mathbb{H}_2$. If on the other hand there is a transversal T of $\mathbb{H}_1$ which is an independent set of $\mathbb{H}_2$, then the assignment $\varphi := \langle v \to 1 : v \in T\rangle \cup \langle v \to 0 : v \in \mathrm{var}(F) \setminus T\rangle$ is a satisfying assignment for F. □

Now a pair $((\mathbb{H}_1, \mathbb{H}_2), \zeta)$, where $(\mathbb{H}_1, \mathbb{H}_2)$ is a PN-pair and $\zeta : \mathcal{VA} \to \{-1, +1\}$ is a sign-flip, *represents* a clause-set F if $F' = \zeta * F(\mathbb{H}_1, \mathbb{H}_2)$, where "$*$" denotes the application of the sign flip, and F' is obtained from F by removal of pure literals (setting them to false, not to true(!)). If a clause-set can be represented in this way, then it is bipartite. In the reverse direction we have

Lemma 6. *For bipartite clause-sets a representation can be computed in quadratic time.*

Since we want to apply the algorithm of Lemma 6 to some subclasses of the class of bipartite clause-sets, the following notion will be useful. Given a class $\mathcal{C}$ of bipartite clause-sets and a class $\mathcal{H}$ of PN-pairs, we say that $\mathcal{H}$ *represents* $\mathcal{C}$ if for all $F \in \mathcal{CLS}$ the following statements are equivalent:

1. $F \in \mathcal{C}$;
2. for all representations $((\mathbb{H}_1, \mathbb{H}_2), \zeta)$ of F we have $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{H}$;
3. there exists a representation $((\mathbb{H}_1, \mathbb{H}_2), \zeta)$ of F with $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{H}$.

4.2 Bi-hitting clause-sets and bi-hitting PN-pairs

A clause-set $F \in \mathcal{CLS}$ is called a **bi-hitting clause-set** if the conflict multigraph of F is complete bipartite, or in other words, if there exists a permutation matrix P and a matrix B with no zero-entry such that $\mathrm{C_s}(F) = P^t \cdot \left(\begin{smallmatrix} 0 & B \\ B^t & 0 \end{smallmatrix}\right) \cdot P$. As before, where we used PN-pairs as the "essential" representations of bipartite clause-sets, we now consider "bi-hitting PN-pairs" as the essential representations of bi-hitting clause-sets. A **bi-hitting PN-pair** is a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ such that every $H \in \mathbb{H}_2$ is a transversal of $\mathbb{H}_1$. If $(\mathbb{H}_1, \mathbb{H}_2)$ is a bi-hitting PN-pair, then so is $(\mathbb{H}_2, \mathbb{H}_1)$.

Lemma 7. *The class of bi-hitting PN-pairs represents the class of bi-hitting clause-sets.*

A **transversal PN-pair** is a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ with $\mathbb{H}_2 = \mathrm{Tr}(\mathbb{H}_1)$ and $\mathbb{H}_1 = \mathrm{Tr}(\mathbb{H}_2)$. Every transversal PN-pair is a bi-hitting PN-pair, and if $(\mathbb{H}_1, \mathbb{H}_2)$ is a transversal PN-pair, then so is $(\mathbb{H}_2, \mathbb{H}_1)$. We want to recognise transversal PN-pair as the representations of unsatisfiable bi-hitting clause-sets, and to do so, we need to remove subsumed clauses. A **simple PN-pair** is a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ such that $\mathbb{H}_1$ and $\mathbb{H}_2$ are simple hypergraphs, that is, they do not contain subsumed hyperedges. Every transversal PN-pair is simple. If $(\mathbb{H}_1, \mathbb{H}_2)$ is a simple PN-pair, then $(\mathbb{H}_1, \mathbb{H}_2)$ is a transversal PN-pair iff $\mathbb{H}_2 = \mathrm{Tr}(\mathbb{H}_1)$.

Lemma 8. *Consider a simple bi-hitting PN-pair* $(\mathbb{H}_1, \mathbb{H}_2)$*. Then* $F(\mathbb{H}_1, \mathbb{H}_2)$ *is unsatisfiable if and only if* $(\mathbb{H}_1, \mathbb{H}_2)$ *is a transversal PN-pair.*

Proof. First assume $F(\mathbb{H}_1, \mathbb{H}_2)$ is unsatisfiable. We have to show that $\mathrm{Tr}(\mathbb{H}_1) = \mathbb{H}_2$. By the definition of bi-hitting every $T \in \mathbb{H}_2$ is a transversal of $\mathbb{H}_1$. If there would exist a transversal T' of $\mathbb{H}_1$ with $T' \subset T$, then by Lemma 5 there would exist some $T'' \in \mathbb{H}_2$ with $T'' \subseteq T' \subset T$ contradicting the simplicity of $\mathbb{H}_2$. Thus $\mathbb{H}_2 \subseteq \mathrm{Tr}(\mathbb{H}_1)$, and by Lemma 5 in fact equality holds. So we have shown that $(\mathbb{H}_1, \mathbb{H}_2)$ is a transversal PN-pair. If on the other hand $(\mathbb{H}_1, \mathbb{H}_2)$ is a transversal PN-pair, then immediately by Lemma 5 unsatisfiability of $F(\mathbb{H}_1, \mathbb{H}_2)$ follows. □

The decision problem whether a PN-pair is a transversal PN-pair is known in the literature under the name of the *hypergraph transversal problem.* In [7] it has been shown that the hypergraph transversal problem is decidable in quasi-polynomial time (more precisely in time $O(s^{o(\log s)})$, where s is the sum of the sizes of the two hypergraphs). Thus by lemmata 6, 7 and 8 we get

Theorem 9. *The satisfiability problem for the class of bi-hitting clause-sets is solvable in quasi-polynomial time (more precisely in time* $O(\ell(F)^{o(\log \ell(F))})$ *for* $F \in \mathcal{CLS}$*, where* $\ell(F) := \sum_{C \in F} |C|$ *is the number of literal occurrences in* F*).*

It is an important open problem whether the hypergraph transversal problem is decidable in polynomial time, which is equivalent to the problem, whether the satisfiability problem for the class of bi-hitting clause-sets is decidable in polynomial time.

4.3 Uniform bi-hitting clause-sets and exact bi-hitting PN-pairs

We have seen in Theorem 9 that satisfiability for clause-sets with complete bipartite conflict multigraph and thus also for clause-sets in $\mathcal{CLS}_h(1)$ (see Theorem 3) is decidable in quasi-polynomial time. Now we will look at the case where the conflict multigraph of a clause-set $F \in \mathcal{CLS}$ is a complete bipartite *graph*, which is equivalent to the existence of a permutation matrix P and a matrix J with all entries equal to 1 such that $\mathrm{C_s}(F) = P^t \cdot \left(\begin{smallmatrix} 0 & J \\ J^t & 0 \end{smallmatrix}\right) \cdot P$. We will show polynomial time satisfiability decision for this class.

$F \in \mathcal{CLS}$ is called a k**-uniform bi-hitting clause-set** for $k \geq 0$ if $\mathrm{C_s}(F) = P^t \cdot \left(\begin{smallmatrix} 0 & k\cdot J \\ k\cdot J^t & 0 \end{smallmatrix}\right) \cdot P$, while **uniform bi-hitting** means k-uniform bi-hitting for some k. A k-uniform bi-hitting clause-set F for $k \geq 2$ is unsatisfiable iff $\bot \in F$, and thus in the remainder we consider only the case $k = 1$. A transversal T of a hypergraph $\mathbb{H}$ is called *exact* ([5]) if for all $H \in \mathbb{H}$ we have $|T \cap H| = 1$. And $\mathbb{H}$ is called *exact* if every minimal transversal of $\mathbb{H}$ is exact. An **exact bi-hitting PN-pair** is a PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ such that every $H \in \mathbb{H}_2$ is an exact transversal of $\mathbb{H}_1$. Every exact bi-hitting PN-pair is a bi-hitting PN-pair, and if $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact bi-hitting PN-pair, then so is $(\mathbb{H}_2, \mathbb{H}_1)$.

Lemma 10. *Exact bi-hitting PN-pairs represent* 1*-uniform bi-hitting clause-sets.*

An **exact transversal PN-pair** is a transversal PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ where $\mathbb{H}_1, \mathbb{H}_2$ are exact hypergraphs. If $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact transversal PN-pair, then so is $(\mathbb{H}_2, \mathbb{H}_1)$. If $(\mathbb{H}_1, \mathbb{H}_2)$ is a simple PN-pair, then $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact transversal PN-pair iff $\mathbb{H}_2 = \mathrm{Tr}(\mathbb{H}_1)$ and $\mathbb{H}_1$ is exact. Since a transversal PN-pair $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact transversal PN-pair iff $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact bi-hitting PN-pair, immediately from Lemma 8 we get

Lemma 11. *Consider a simple exact bi-hitting PN-pair* $(\mathbb{H}_1, \mathbb{H}_2)$*. Then the clause-set* $F(\mathbb{H}_1, \mathbb{H}_2)$ *is unsatisfiable if and only if* $(\mathbb{H}_1, \mathbb{H}_2)$ *is an exact transversal PN-pair.*

The *exact transversal hypergraph problem* is the problem to decide, whether a given PN-pair is an exact transversal PN-pair. In [5] it is shown that the exact transversal hypergraph problem can be decided in polynomial time, and thus we get

Theorem 12. *The SAT problem for uniform bi-hitting clause-sets is decidable in polynomial time.*

By Theorem 3 modulo blocked clauses the class of clause-sets with hermitian rank at most one is included in the class of 1-uniform bi-hitting clause-sets, and thus

Theorem 13. *SAT decision for the class* $\mathcal{CLS}_h(1)$ *can be done in polynomial time.*

We conclude this section by reviewing the proof of [5] (Theorem 3.3), that the decision problem is solvable in polynomial time, whether a simple hypergraph $\mathbb{H}$ is an "exact transversal hypergraph", that is, whether $(\mathbb{H}, \mathrm{Tr}(\mathbb{H}))$ is an exact transversal PN-pair. This together with Theorem 4.3 of [5], that for exact transversal hypergraphs the set of minimal transversals can be enumerated with polynomial delay (i.e., the time between two consecutive outputs is polynomially bounded in the size of the input), implies Theorem 12.

Let $\mathbb{H}(v) := \{H \in \mathbb{H} : v \in H\}$ be the star of vertex v, and let $V(\mathbb{H}) := \bigcup \mathbb{H} = \bigcup_{H \in \mathbb{H}} H$ denote the vertex set of $\mathbb{H}$. First we make the observation that a transversal T of a hypergraph $\mathbb{H} \neq \emptyset$ is minimal iff there exists $H \in \mathbb{H}$ with $|T \cap H| = 1$. Now it follows easily that $\mathbb{H}$ is exact transversal if and only if for all vertices $v \in V(\mathbb{H})$, for all hyperedges $H \in \mathbb{H}(v)$ and for all transversals $T \in \mathrm{Tr}(\mathbb{H})$ in case of $v \in T$ and $T \cap H = \{v\}$ if follows $T \cap V(\mathbb{H}(v)) = \{v\}$. This condition is equivalent to

$$\forall\, v \in V(\mathbb{H})\ \forall\, H \in \mathbb{H}(v) : V(\mathbb{H}(v)) \cap V(\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T \wedge T \cap H = \{v\}\}) = \{v\}. \qquad (1)$$

For any hypergraph $\mathbb{H}$ by definition we have

$$\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T \wedge T \cap H = \{v\}\} = \big\{\, T \cup \{v\} : T \in \mathrm{Tr}(\{H' \setminus H : H' \in \mathbb{H} \wedge v \notin H'\}) \,\big\},$$

and thus (1) is equivalent to

$$\forall\, v \in V(\mathbb{H})\ \forall\, H \in \mathbb{H}(v) : V(\mathbb{H}(v)) \cap V(\mathrm{Tr}(\{H' \setminus H : H' \in \mathbb{H} \wedge v \notin H'\})) = \emptyset. \qquad (2)$$

Using $\min(\mathbb{H})$ for the set of inclusion-minimal elements of a hypergraph $\mathbb{H}$, the trick is now to exploit the observation $V(\mathrm{Tr}(\mathbb{H})) = V(\min(\mathbb{H}))$ for any hypergraph $\mathbb{H}$, which follows from $\mathrm{Tr}(\mathrm{Tr}(\mathbb{H})) = \min(\mathbb{H})$, and which yields, that (2) is equivalent to

$$\forall\, v \in V(\mathbb{H})\ \forall\, H \in \mathbb{H}(v) : V(\mathbb{H}(v)) \cap V(\min(\{H' \setminus H : H' \in \mathbb{H} \wedge v \notin H'\})) = \emptyset,$$

where this final criterion obviously is decidable in polynomial time. The idea of this nice proof can be motivated as follows: By definition $\mathbb{H}$ is exact transversal iff for all $v \in V(\mathbb{H})$ we have

$$V(\mathbb{H}(v)) \cap V(\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T\}) = \{v\}.$$

The problem is to compute $V(\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T\})$, where we would like to recognise $\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T\}$ as the transversal hypergraph of some $\mathbb{H}'$, which would yield $V(\{T \in \mathrm{Tr}(\mathbb{H}) : v \in T\}) = V(\min(\mathbb{H}')$, avoiding the computation of the transversal hypergraph. Selecting exactly the minimal transversals T of $\mathbb{H}$ containing v seems not possible, but if we fix a hyperedge $H \in \mathbb{H}$ with $v \in H$, then the minimal transversals of $\mathbb{H}$ containing v but not any other vertex from H are exactly the $T \cup \{v\}$, where T is a minimal transversals of the hyperedges of $\mathbb{H}$ not containing v with all other vertices from H removed, and we arrive at the above proof.

5 Open problems

5.1 Polynomial time SAT decision for bounded hermitian rank

As mentioned in the introduction, it would be interesting to prove an upper bound $2^{h(F)}$ on time complexity of satisfiability decision (ignoring polynomial factors). We would achieve this aim, if we can find a polynomial time reduction $r : \mathcal{CLS} \to \mathcal{CLS}$ and some class $\mathbb{E} \subseteq \mathcal{CLS}$ which is decidable and satisfiability decidable in polynomial time, such that for all clause-sets $F \in \mathcal{CLS}$ and $F' := r(F)$ we have $h(F') \le h(F)$, and we have $F' \in \mathbb{E}$ or there exists a variable $v \in \mathrm{var}(F')$ such that for both truth values $\varepsilon \in \{0,1\}$ we have $h(\langle v \to \varepsilon\rangle * F') < h(F')$. For $r = \mathrm{id}_{\mathcal{CLS}}$ and $\mathbb{E} = \mathcal{CLS}_h(1)$ this property does not hold.

Since application of partial assignments does not increase the hermitian rank, when allowing a logarithmic factor in the exponent it actually suffices to find a variable $v \in \mathrm{var}(F)$ such that for just *one truth value* $\varepsilon \in \{0,1\}$ we have $h(\langle v \to \varepsilon\rangle * F) < h(F)$ (following [10, 16]). Using computer experiments, we did not find a counterexample of small dimension, and so we conjecture

Conjecture 14. For all $F \in \mathcal{CLS}$ with $h(F) \ge 2$ there exists $v \in \mathrm{var}(F)$ and $\varepsilon \in \{0,1\}$ with $h(\langle v \to \varepsilon\rangle * F) < h(F)$.

If Conjecture 14 is true, then by Lemma 3.7 in [10] or Theorem 4.3 in [16] we get satisfiability decision for $\mathcal{CLS}$ in time $n(F)^{2h(F)}$ (ignoring polynomial factors). And furthermore the hardness $h_{\mathcal{CLS}_h(1)}(F)$ of clause-sets F as studied in [10, 16], using the polynomial time satisfiability decision for $\mathcal{CLS}_h(1)$ as oracle, would be bounded by $h_{\mathcal{CLS}_h(1)}(F) \le h(F) - 1$. Obviously, Conjecture 14 implies polynomial time satisfiability decision for formulas with bounded hermitian rank, and thus

Conjecture 15. For fixed $k \ge 0$ satisfiability decision of the class $\mathcal{CLS}_h(k)$ can be done in polynomial time.

5.2 Characterising (minimally) unsatisfiable uniform bi-hitting clause-sets

We now have a look at the bearings of the investigations of this paper on minimally unsatisfiable clause-sets (all proofs can be found in [15]). Every unsatisfiable clause-set $F \in \mathcal{USAT}$ contains some minimally unsatisfiable sub-clause-set $F' \in \mathcal{MUSAT}$. Let us say that F has a *unique core*, if there is exactly one $F' \subseteq F$ with $F' \in \mathcal{MUSAT}$, in which case we call F' the *core* of F. Related to the formula classes considered in this article are three classes of clause-sets with unique core:

1. Every unsatisfiable bi-hitting clause-set F has a unique core given by the set of subsumption-minimal clauses of F, obtained from F by elimination of subsumed clauses.

2. For unsatisfiable uniform bi-hitting clause-sets the unique core can also be obtained by elimination of pure literals (i.e,, the unique core is the lean kernel w.r.t. pure autarkies).
3. Every unsatisfiable clause-set F with $h(F) \leq 1$ and $\bot \notin F$ has a unique core, obtained from F by elimination of blocked clauses.

The class of cores of unsatisfiable bi-hitting clause-sets, the set of minimally unsatisfiable bi-hitting clause-sets, is the set of clause-sets which can be represented by transversal PN-pairs. Let $\mathcal{TPN}$ be the set of transversal PN-pairs, and for any set $\mathcal{P}$ of PN-pairs let $\mathcal{CLS}(\mathcal{P})$ denote the set of clause-sets $F(\mathbb{H}_1, \mathbb{H}_2)$ for some $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{P}$. Thus $\mathcal{CLS}(\mathcal{TPN})$ is contained in the set of minimally unsatisfiable bi-hitting clause-sets, and for every minimally unsatisfiable bi-hitting clause-set F there exists a sign-flip ζ with $\zeta * F \in \mathcal{CLS}(\mathcal{TPN})$. Following [3], a *splitting* of $F \in \mathcal{MUSAT}$ on a variable $v \in \mathrm{var}(F)$ is a pair $(F_0, F_1) \in \mathcal{MUSAT}^2$ with $F_\varepsilon \subseteq \langle v \to \varepsilon \rangle * F$ for $\varepsilon \in \{0, 1\}$. A class $\mathcal{C} \subseteq \mathcal{MUSAT}$ is called *closed under splitting* if for every $F \in \mathcal{C}$ and every splitting (F_0, F_1) of F we have $F_0, F_1 \in \mathcal{C}$. It is $\mathcal{CLS}(\mathcal{TPN})$ closed under splitting, and every element of $\mathcal{CLS}(\mathcal{TPN})$ has a unique splitting (the same holds for the larger class of minimally unsatisfiable bi-hitting clause-sets). Characterising minimally unsatisfiable bi-hitting clause-sets amounts now to characterise $\mathcal{TPN}$, which seems to be an elusive task, so we have to consider simpler cases.

The class of cores of unsatisfiable uniform bi-hitting clause-sets is identical to the class of cores of unsatisfiable clause-sets with hermitian rank at most one, and can be described as the class of clause-sets representable by some $((\mathbb{H}_1, \mathbb{H}_2), \zeta)$, where $(\mathbb{H}_1, \mathbb{H}_2)$ is an exact transversal PN-pair. Let $\mathcal{ETPN}$ be the set of exact transversal PN-pairs. It is $\mathcal{CLS}(\mathcal{ETPN}) \subset \mathcal{CLS}(\mathcal{TPN})$ contained in the set of minimally unsatisfiable uniform bi-hitting clause-sets, and for every minimally unsatisfiable uniform bi-hitting clause-set F there exists a sign-flip ζ with $\zeta * F \in \mathcal{CLS}(\mathcal{ETPN})$. The class $\mathcal{CLS}(\mathcal{ETPN})$ is again closed under splitting (as is the larger class of minimally unsatisfiable uniform bi-hitting clause-sets). Characterising the class of minimally unsatisfiable uniform bi-hitting clause-sets amounts to characterising $\mathcal{ETPN}$. We can construct elements of $\mathcal{ETPN}$ as follows:

1. For variables $v_1, \dots, v_n$, $n \in \mathbb{N}_0$ we have $(\{\{v_1, \dots, v_n\}\}, \{\{v_i\} : i \in \{1, \dots, n\}\}) \in \mathcal{ETPN}$.
2. If $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{ETPN}$, then also $(\mathbb{H}_2, \mathbb{H}_1) \in \mathcal{ETPN}$.
3. If $(\mathbb{H}_1, \mathbb{H}_2), (\mathbb{H}'_1, \mathbb{H}'_2) \in \mathcal{ETPN}$ with $\emptyset \notin \mathbb{H}_1 \cup \mathbb{H}'_1$ and $V(\mathbb{H}_1) \cap V(\mathbb{H}'_1) = \emptyset$, then
$$(\mathbb{H}_1 \cup \mathbb{H}'_1, \{T \cup T' : T \in \mathbb{H}_2 \wedge T' \in \mathbb{H}'_2\}) \in \mathcal{ETPN}.$$

Conjecture 16. $\mathcal{ETPN}_0 = \mathcal{ETPN}$, where $\mathcal{ETPN}_0$ is the set of exact transversal PN-pairs created by these rules.

If Conjecture 16 is true, then we can draw the following conclusions:

1. We get a more efficient satisfiability decision procedure for the class of uniform bi-hitting clause-sets (and for $\mathcal{CLS}_h(1)$) than the one outlined at the end of Subsection 4.3.
2. The elements of $\mathcal{CLS}(\mathcal{ETPN}_0)$ have a read-once resolution refutation (the simplest possible resolution refutations: a tree-resolution refutation where every node is labelled with a unique clause), and thus in fact every unsatisfiable uniform bi-hitting clause-set (as well as every element of $\mathcal{CLS}_h(1) \cap \mathcal{USAT}$) would have a read-once resolution refutation.

As a partial result towards Conjecture 16 we can completely characterise exact transversal PN-pairs $(\mathbb{H}_1, \mathbb{H}_2)$, where the rank of $\mathbb{H}_1$ or $\mathbb{H}_2$ is at most 2 (i.e., there is $i \in \{1, 2\}$ such that for all $H \in \mathbb{H}_i$ we have $|H| \leq 2$). Consider a simple hypergraph $\mathbb{H}$. Call $\mathbb{H}$ *exact transversal* if $(\mathbb{H}, \mathrm{Tr}(\mathbb{H}))$ is an exact transversal PN-pair. If $\mathbb{H} = \emptyset$, then $\mathbb{H}$ is exact transversal. If $\emptyset \in \mathbb{H}$, then $\mathbb{H} = \{\emptyset\}$ and $\mathbb{H}$ is exact transversal. If $\{x\} \in \mathbb{H}$, then $\mathbb{H}$ is exact transversal iff $\mathbb{H} \setminus \{x\}$ is exact transversal. So w.l.o.g. we assume that $\mathbb{H}$ is not empty and the size of a smallest hyperedge is at least 2.

Assume furthermore that the rank of $\mathbb{H}$ is 2. Now $\mathbb{H}$ constitutes a graph. And if $\mathbb{H}$ as a graph is connected, then $\mathbb{H}$ is exact transversal if and only if $\mathbb{H}$ as a graph is complete bipartite. Complete bipartite graphs with at least two vertices are exactly the transversal hypergraphs of hypergraphs $\{A, B\}$ for non-empty disjoint A, B. It follows that for an exact transversal PN-pairs $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{ETPN}$ with rank of $\mathbb{H}_1$ or $\mathbb{H}_2$ at most 2 we have $(\mathbb{H}_1, \mathbb{H}_2) \in \mathcal{ETPN}_0$.

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