Computing Unsatisfiable k-SAT Instances with Few Occurrences per Variable*

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Abstract. (k,s)-SAT is the propositional satisfiability problem restricted to instances where each clause has exactly k distinct literals and every variable occurs at most s times. It is known that there exists an exponential function f such that for $s \leq f(k)$ all (k,s)-SAT instances are satisfiable, but $(k, f(k)+1)$-SAT is already NP-complete ($k \geq 3$). Exact values of f are only known for $k = 3$ and $k = 4$, and it is open whether f is computable.
We introduce a computable function f_1 which bounds f from above and determine the values of f_1 by means of a calculus of integer sequences. This new approach enables us to improve the best known upper bounds for $f(k)$, generalizing the known constructions for unsatisfiable (k,s)-SAT instances for small k.

1 Introduction

We consider CNF formulas represented as sets of clauses. Let k, s be fixed positive integers. We denote by (k,s)-CNF the set of formulas F where every clause of F has *exactly* k different literals and each variable occurs in *at most* s clauses of F. We denote the sets of satisfiable and unsatisfiable formulas by SAT and UNSAT, respectively.

It was observed by Tovey [1] that all formulas in $(3,3)$-CNF are satisfiable, and the satisfiability problem restricted to $(3,4)$-CNF is already NP-complete. This was generalized in Kratochvíl, et al. [2] where it is shown that for every $k \geq 3$ there is some integer $s = f(k)$ such that

1. all formulas in (k,s)-CNF are satisfiable, and
2. $(k, s+1)$-SAT, the SAT problem restricted to $(k, s+1)$-CNF, is already NP-complete.

The function f can be defined by the equation

$$f(k) := \max\{\, s : (k,s)\text{-CNF} \cap \text{UNSAT} = \emptyset \,\}.$$

From [1] it follows that $f(3) = 3$ and $f(k) \geq k$ for $k > 3$.

Asymptotic upper and lower bounds for $f(k)$ have been obtained in [2,3,4]. Since typical formulas arising in practice have clauses of small width, it is interesting to know the exact values of $f(k)$ for small k. However, it is not known whether f is computable.

Dubois [5] constructs unsatisfiable formulas in $(4,6)$-CNF and $(5,11)$-CNF, respectively, which implies $4 \leq f(4) \leq 5$ and $5 \leq f(5) \leq 10$. As reported in [3], Stříbrná shows in her M.Sc. thesis [6] that $(4,5)$-CNF contains unsatisfiable formulas, hence $f(4) = 4$. More recently, Berman, et al. [7] construct unsatisfiable formulas belonging to the classes $(3,4)$-CNF, $(4,6)$-CNF, $(5,9)$-CNF, improving Dubois' upper bound for $f(5)$ to 8.

The quoted constructions are quite involved. We present a new and simple technique for generating unsatisfiable (k,s)-CNF formulas. By this new technique we can improve on best known upper bounds for $f(k)$; Table 1 gives an overview of upper bounds for $f(k)$.

By means of a construction due to Kratochvíl, et al. [2], one can construct from any unsatisfiable (k,s)-CNF formula an unsatisfiable $(k+1, 2s)$-CNF formula; thus

$$f(k+1) \leq 2f(k) + 1. \tag{1}$$

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By generalization of a theorem by Savický and Sgall [3] one can derive the equation

$$f(3k) \leq 3 \cdot 4^{k-1} f(k), \tag{2}$$

which yields asymptotically better upper bounds for $f(k)$; for small k, however, (1) is preferable. The upper bounds for $f(k)$ obtained by (1) are still relatively large compared to upper bounds obtained by genuinely constructed formulas (see also Table 1).

	Tov[1]	Dub[5]	Stř[6]	BKS[7]	this paper
$3 \leq f(3) \leq$	3	3	3	3	3
$4 \leq f(4) \leq$	7*	5	4	5	4
$5 \leq f(5) \leq$	15*	10	9*	8	7
$7 \leq f(6) \leq$	31*	21*	19*	17*	11
$13 \leq f(7) \leq$	63*	43*	39*	35*	17
$24 \leq f(8) \leq$	127*	87*	79*	71*	29
$41 \leq f(9) \leq$	255*	175*	159*	143*	51

Table 1. Best known lower and upper bounds of $f(k)$ for small k. Entries labeled by an asterisk are obtained via equation (1) from the preceding value of the respective paper. The lower bounds are taken from [7].

Our approach is to focus on a certain class MU(1) of unsatisfiable formulas. Formulas in MU(1) have a simple structure and can be constructed in a recursive way (see the next section). Therefore it is easier to search for unsatisfiable formulas in (k,s)-CNF $\cap$ MU(1) than in (k,s)-CNF.

For $k \geq 3$ let $f_1(k)$ denote the largest integer such that (k,s)-CNF $\cap$ MU(1) $= \emptyset$. Since all formulas in MU(1) are unsatisfiable, always $f(k) \leq f_1(k)$ holds. Our examples below show that $f(k) = f_1(k)$ for $k = 3, 4$. It is interesting to know whether $f(k) = f_1(k)$ holds for $k \geq 5$.

By our new approach, the construction of unsatisfiable (k,s)-CNF formulas can be reduced to applying a certain operation to ordered integer sequences. Therefore, the construction can be easily automatized (a saturation algorithm is outlined below). The next theorem summarizes the results we have obtained so far by running a C++ implementation of the saturation algorithm.

Theorem 1
The following classes contain unsatisfiable formulas: $(3,4)$-CNF, $(4,5)$-CNF, $(5,8)$-CNF, $(6,12)$-CNF, $(7,18)$-CNF, $(8,30)$-CNF. $(9,52)$-CNF. *Hence, the satisfiability problem restricted to any of these classes is* NP*-complete.*

The existence of unsatisfiable formulas in $(5,8)$-CNF and $(6,12)$-CNF is certified by the derivations given in Fig. 3 and the appendix, respectively. For the other classes mentioned in Theorem 1, certificates can be found in a file archive, available at the authors' homepages.

2 The Class MU(1)

A CNF formula is *minimal unsatisfiable* if it is unsatisfiable and removing any of its clauses makes it satisfiable. We denote the class of minimal unsatisfiable CNF formulas by MU. Since every unsatisfiable formula F has a minimal unsatisfiable subset F', and since $F \in (k,s)$-CNF implies $F' \in (k,s)$-CNF, we can restrict ourselves to the class MU. In other words,

$$f(k) = \max\{\, s : (k,s)\text{-CNF} \cap \text{MU} = \emptyset \,\}.$$

The *deficiency* $\delta(F)$ of a formula with n variables and m clauses is defined as $\delta(F) = m - n$. It is known that formulas in MU have always positive deficiency [8]; therefore it is natural to parameterize MU by deficiency and to consider the classes $\text{MU}(k) := \{\, F \in \text{MU} : \delta(F) = k \,\}$ for $k \geq 1$.

Let us consider the function

$$f_1(k) = \max\{\, s : (k,s)\text{-CNF} \cap \text{MU}(1) = \emptyset \,\}. \tag{3}$$

Evidently, we have $f_1(k) \geq f(k)$, and so any upper bound for $f_1(k)$ is also an upper bound for $f(k)$. In the sequel we will show that f_1 is computable, and that for small k we can actually compute the exact value of $f_1(k)$.

Formulas in MU(1) have been widely studied (see, e.g., [8,9,10,11,12]). In particular, the following result of Davydov, et al. [9] (a proof is implicitly present in [8]), shows that formulas in MU(1) can be recursively decomposed ($\mathsf{var}(F)$ denotes the set of variables which occur (positively or negatively) in the formula F).

Lemma 1 (Davydov, et al. [9]) *$F \in \text{MU}(1)$ if and only if either $F = \{\emptyset\}$ or F is the disjoint union of formulas F_1', F_2' such that for a variable x we have*

- $\mathsf{var}(F_1') \cap \mathsf{var}(F_2') = \{x\}$ *and* $\{x, \overline{x}\} \subseteq \bigcup_{C \in F} C$;
- $F_1 := \{\, C \setminus \{x\} : C \in F_1' \,\} \in \text{MU}(1)$;
- $F_2 := \{\, C \setminus \{\overline{x}\} : C \in F_2' \,\} \in \text{MU}(1)$.

If F has a variable x with the properties stated in the above lemma, then following [11] we call the pair (F_1, F_2) a *disjunctive splitting of F in x*. Furthermore we call the number of clauses of F in which x occurs the *degree* of the splitting (F_1, F_2).

For example, the formula $F = \{\{x,z\},\{\overline{x},y\},\{\overline{y},z\},\{\overline{z},w\},\{\overline{z},\overline{w}\}\}$ belongs to MU(1) since it can be decomposed by disjunctive splittings as displayed in Fig. 1. Note that $F \in (2,4)$-CNF since all clauses have size 2 and every literal occurs at most 4 times. In general, if we decompose a formula F by splittings of degree $\leq s$, then evidently every variable of F occurs in at most s clauses. Hence we have the following lemma.

$$\dfrac{\dfrac{\dfrac{\{\emptyset\} \qquad \{\emptyset\}}{\{\{x\},\{\overline{x}\}\}}\ (\text{split in } x) \qquad \{\emptyset\}}{\{\{x\},\{\overline{x},y\},\{\overline{y}\}\})}\ (\text{split in } y) \qquad \dfrac{\{\emptyset\} \qquad \{\emptyset\}}{\{\{w\},\{\overline{w}\}\}}\ (\text{split in } w)}{F = \{\{x,z\},\{\overline{x},y\},\{\overline{y},z\},\{\overline{z},w\},\{\overline{z},\overline{w}\}\}}\ (\text{split in } z)$$

Fig. 1. Decomposition of a formula $F \in \text{MU}(1)$ by disjunctive splittings.

Lemma 2 *If all clauses of a nonempty formula F have size k, then $F \in (k,s)$-CNF $\cap$ MU(1) if and only if F can be decomposed by disjunctive splittings of degree $\leq s$.*

3 A Calculus of Integer Sequences

Let $\sigma = (a_1, \ldots, a_n)$ be a *finite nonincreasing sequence of positive integers* (a *stairway*, for short). That is, $a_1 \geq \cdots \geq a_n \geq 1$. We call a_i an *entry* of σ, n the *length* of σ, and denote the *empty sequence* by ε. For a finite sequence of non-negative integers σ let σ^{ord} denote the stairway obtained from σ by removing 0's and by ordering the entries nonincreasingly.

For a fixed integer $s \geq 2$ we consider the (nondeterministic) binary rule $N(s)$ that allows to infer a stairway σ from stairways σ_1, σ_2 as follows: For $i = 1,2$ obtain σ_i' from σ_i by decrementing $s_i \geq 1$ entries, $s_1 + s_2 \leq s$, and put $\sigma := (\sigma_1'\sigma_2')^{\text{ord}}$. An *$N(s)$-derivation* is a finite binary rooted tree T whose vertices are labeled by stairways such that if a vertex v labeled by σ has parents v_1, v_2 labeled by σ_1, σ_2, respectively, then σ can be *inferred* from σ_1, σ_2 by the rule $N(s)$. For a set of stairways Γ and a stairway σ we write $\Gamma \vdash_{N(s)} \sigma$ if there is an $N(s)$-derivation T whose root is

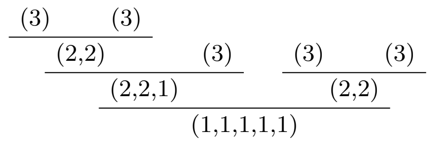

Fig. 2. An $N(4)$-derivation.

labeled by σ and whose leaves are labeled by sequences from Γ. In particular, we have $\Gamma \vdash_{N(s)} \sigma$ if $\sigma \in \Gamma$. If Γ is a singleton $\{\sigma'\}$ we simply write $\sigma' \vdash_{N(s)} \sigma$.

As an example, the $N(4)$-derivation displayed in Fig. 2 shows that $(3) \vdash_{N(4)} (1,1,1,1,1)$.

Let $F = \{C_1, \dots, C_m\} \neq \emptyset$ be a formula with $0 \leq |C_1| \leq \cdots \leq |C_m| \leq k$, and let n be the largest integer in $\{1, \dots, m\}$ with $|C_n| < k$. We associate with F the stairway

$$\Sigma_k(F) := (k - |C_1|, \dots, k - |C_n|).$$

For dealing formally with the rule $N(s)$ in the proofs below, the following concept is convenient. Consider stairways $\sigma_1 = (a_1, \dots, a_j)$ and $\sigma_2 = (a_{j+1}, \dots, a_m)$. The definition of $N(s)$ says that a stairway σ can be inferred from σ_1, σ_2 if and only if there is a set $I \subseteq \{1, \dots, m\}$ with $I \cap \{1, \dots, j\} \neq \emptyset$, $I \cap \{j+1, \dots, m\} \neq \emptyset$, and $|I| \leq s$ such that $\sigma = (a'_1, \dots, a'_m)^{\mathrm{ord}}$ where

$$a'_i = \begin{cases} a_i - 1 & \text{if } i \in I; \\ a_i & \text{otherwise.} \end{cases}$$

We call the set I an *index set* associated with the inference. Note that the index set I is not necessarily unique.

The next lemma, which can be shown by induction, asserts that $N(s)$-derivations and formulas in $\mathrm{MU}(1) \cap (k,s)$-CNF are closely related.

Lemma 3 *For every stairway σ the following holds true. $(k) \vdash_{N(s)} \sigma$ if and only if there is a formula $F \in \mathrm{MU}(1)$ with $\Sigma_k(F) = \sigma$ which can be decomposed by disjunctive splittings of degree $\leq s$.*

Proof. ($\Rightarrow$) Assume $(k) \vdash_{N(s)} \sigma$ and let T be an $N(s)$-derivation of σ from (k) with a minimal number n of inference steps (we count every non-leaf of T as an inference step). We proceed by induction on n. If $n = 0$ then σ is the axiom (k) and we put $F = \{\emptyset\}$. Clearly $\Sigma_k(F) = (k)$ and we are done. Now assume $n \geq 1$, and let σ_1, σ_2 be the stairways from which σ is inferred in T. Let $\sigma_1 = (a_1, \dots, a_j)$, $\sigma_2 = (a_{j+1}, \dots, a_m)$, and $\sigma = (c_1, \dots, c_n)$. Let $I \subseteq \{1, \dots, m\}$ be an index set associated with the inference of σ from σ_1, σ_2, so that we can write $\sigma = (a'_1, \dots, a'_m)^{\mathrm{ord}}$.

By induction hypothesis (the subderivations of T ending in σ_1 and σ_2, respectively, have less than n steps), there are formulas $F_1, F_2 \in \mathrm{MU}(1)$ with $\Sigma_k(F_i) = \sigma_i$ such that F_i can be decomposed by disjunctive splittings of degree $\leq s$. We may assume that F_1 and F_2 do not share a variable (we can always rename variables). Let F'_i be the subset of F_i containing all clauses of size k, $i = 1, 2$. Since $\Sigma_k(F_i) = \sigma_i$, we can write $F_1 = \{C_1, \dots, C_j\} \cup F'_1$ and $F_2 = \{C_{j+1}, \dots, C_m\} \cup F'_2$ such that $a_i = k - |C_i|$ for $i = 1, \dots, m$. We pick a new variable x and define $F := \{D_1, \dots, D_m\} \cup F'_1 \cup F'_2$ where

$$D_i = \begin{cases} C_i \cup \{x\} & \text{if } i \in I \text{ and } i \leq j \\ C_i \cup \{\overline{x}\} & \text{if } i \in I \text{ and } i > j, \\ C_i & \text{otherwise.} \end{cases}$$

Consequently, (F_1, F_2) is a disjunctive splitting of F of degree $\leq s$. Since $\Sigma_k(F) = \sigma$, the first part of the lemma is shown true.

($\Leftarrow$) Let $F \in \mathrm{MU}(1)$, $\Sigma_k(F) = \sigma$, be decomposable by disjunctive splittings of degree $\leq s$. We show by induction on the number n of variables of F that $(k) \vdash_{N(s)} \sigma$. If $n = 0$ then $F = \{\emptyset\}$ and so $\sigma = (k)$; hence $(k) \vdash_{N(s)} \sigma$. Now assume $n > 0$. By assumption, F has a disjunctive splitting (F_1, F_2) of degree $\leq s$. Let $\sigma_i := \Sigma_k(F_i)$, $i = 1, 2$. Since $|\mathsf{var}(F_i)| \leq |\mathsf{var}(F)| - 1$, it follows by induction hypothesis that $(k) \vdash_{N(s)} \sigma_i$, $i = 1, 2$. It remains to show that σ can be inferred from σ_1, σ_2 by the rule $N(s)$.

By definition of a disjunctive splitting, F is the disjoint union of formulas F_1', F_2' such that for a variable x the conditions stated in Lemma 1 are satisfied. Consequently, for some nonempty subsets $G_i \subseteq F_i$, $i = 1, 2$, we have

$$F_1' = \{ C \cup \{x\} : C \in G_1 \} \cup (F_1 \setminus G_1),$$
$$F_2' = \{ C \cup \{\overline{x}\} : C \in G_2 \} \cup (F_2 \setminus G_2).$$

Since the splitting is of degree $\leq s$, $|G_1| + |G_2| \leq s$ follows. Every clause in $G_1 \cup G_2$ corresponds bijectively to an entry a of σ_i which is decreased by one (thus either $a \geq 2$ and $a - 1$ is an entry of σ, or $a = 1$ and $a - 1$ is omitted in σ). The other clauses $C \in F_i \setminus G_i$ with $|C| < k$ correspond bijectively to entries $a = k - |C|$ of σ_i which give rise to entries of σ. Thus σ can indeed be inferred from σ_1, σ_2 by the rule $N(s)$ and so $(k) \vdash_{N(s)} \sigma$ follows. □

Note that in general there are many different formulas corresponding to one $N(s)$-derivation in the sense of Lemma 3.

For the example in Fig. 1, we have $F = \{\{x, z\}, \{\overline{x}, y\}, \{\overline{y}, z\}, \{\overline{z}, w\}, \{\overline{z}, \overline{w}\}\}$ and $\Sigma_3(F) = (1, 1, 1, 1, 1)$. The disjunctive splitting of degree ≤ 4 depicted in Fig. 1 corresponds to the $N(4)$-derivation in Fig. 2 by means of Lemma 3.

An immediate consequence of Lemma 3 is the following characterization of the function f_1 defined in (3). Recall that ε denotes the empty sequence.

Theorem 2 $f_1(k) = \min\{ s : (k) \vdash_{N(s)} \varepsilon \} - 1$.

Proof. Let $s \geq 2$ such that $(k) \vdash_{N(s)} \varepsilon$. By Lemma 3, there exists a formula $F \in \mathrm{MU}(1)$, $\Sigma_k(F) = \varepsilon$, which can be decomposed by splittings of degree $\leq s$. Thus variables of F occur in at most s clauses. Moreover, $\Sigma_k(F) = \varepsilon$ implies that all clauses of F have size k, thus $F \in (k, s)$-CNF follows. Consequently $f_1(k) \leq s - 1$.

Now assume $f_1(k) \geq s$; i.e., $(k, s)\text{-CNF} \cap \mathrm{MU}(1) = \emptyset$. Consequently, no $F \in \mathrm{MU}(1)$ with $\Sigma_k(F) = \varepsilon$ can be decomposed by splittings of degree $\leq s$. By Lemma 3, it follows that $(k) \vdash_{N(s)} \varepsilon$ does not hold. Hence the theorem is shown true. □

4 Computing f_1

The results of the previous section suggest the following saturation algorithm for determining whether $f_1(k) \leq s$ for given k, s:

- Start with the set $\mathcal{S}_0 = \{(k)\}$.
- For $i > 0$, obtain $\mathcal{S}_i$ as the union of $\mathcal{S}_{i-1}$ and the set of all sequences σ which can be inferred from $\sigma_1, \sigma_2 \in \mathcal{S}_{i-1}$ by the rule $N(s)$.

If we reach a set $\mathcal{S}_i$ which contains the empty sequence ε then we stop, as we then know that $f_1(k) < s$. Otherwise, if we reach a fixed-point i where $\mathcal{S}_i = \mathcal{S}_{i-1}$, then we know $f_1(k) \geq s$. We will show below that a refined saturation algorithm actually terminates, hence that a finite procedure for determining $f_1(k)$ exists.

When we run the saturation algorithm, it is desirable to avoid the derivation of sequences which are "worse" than other already derived sequences. For example, if we have already derived $(3, 2, 1)$, it is certainly superfluous to add the sequence $(3, 3, 1)$ or the sequence $(3, 2, 1, 1)$ to the cumulating set. We will see below that also, say, $(3, 3)$ can be ignored if we already have obtained $(3, 2, 1)$. Formally, we base the comparison of sequences on the following definition.

Let σ, σ' be stairways. We say that σ' is obtained from $\sigma = (a_1, \ldots, a_n)$ by *elementary flattening* if one of the following prevails:

1. For some $p \in \{1, \ldots, n\}$ we have $\sigma' = (a_1', \ldots, a_n')^{\mathrm{ord}}$ where

$$a_i' = \begin{cases} a_i - 1 & \text{if } i = p, \\ a_i & \text{otherwise.} \end{cases}$$

2. Consider σ to have an additional entry a_{n+1} with value 0. For some $p, q \in \{1, \dots, n+1\}$ with $a_p > a_q$ we have $\sigma = (a_1', \dots, a_{n+1}')^{\mathrm{ord}}$ where

$$a_i' = \begin{cases} a_i - 1 & \text{if } i = p, \\ a_i + 1 & \text{if } i = q, \\ a_i & \text{otherwise.} \end{cases}$$

We exclude the case $a_p = a_q + 1$ to ensure $\sigma \neq \sigma'$.

That is, σ' is obtained by decrementing some entry a_p and possibly incrementing some smaller entry a_q. We say that σ' *dominates* σ if either $\sigma' = \sigma$ or σ' can be obtained from σ by multiple applications of elementary flattening.

The next lemma states that if σ is dominated by σ', then σ is "worse" than σ' in the above sense.

Lemma 4 *If σ can be inferred from σ_1, σ_2 by rule $N(s)$, and if σ_i is dominated by $\sigma_i' \neq \varepsilon$, $i = 1, 2$, then σ is dominated by some σ' which can be inferred from σ_1', σ_2' by rule $N(s)$.*

Proof. Since σ_i is dominated by σ_i', σ_i' can be obtained from σ_i by r_i applications of elementary flattening for some $r_i \geq 0$; in symbols, $\sigma_i \xrightarrow{r_i} \sigma_i'$. We proceed by induction on $r = r_1 + r_2$. If $r = 0$ then $\sigma_1 = \sigma_1'$, $\sigma_2 = \sigma_2'$, and we put $\sigma' = \sigma$.

Now assume $r > 0$. W.l.o.g., we may assume that $r_2 > 0$. Hence there is a stairway σ_2^* such that

$$\sigma_2 \xrightarrow{r_2 - 1} \sigma_2^* \xrightarrow{1} \sigma_2'.$$

The induction hypothesis yields that there is a stairway σ^* which dominates σ and can be obtained from σ_1', σ_2^* by the rule $N(s)$. We have to show that there exists a stairway σ' which can be obtained from σ_1', σ_2' by rule $N(s)$ and which dominates σ^*; i.e., that the diagram

$$\begin{array}{ccc} \sigma_1'\sigma_2^* & \xrightarrow{\;\;1\;\;} & \sigma_1'\sigma_2' \\ \Big\downarrow{\scriptstyle N(s)} & & \Big\downarrow{\scriptstyle N(s)} \\ \sigma^* & \xrightarrow{\;\le 1\;} & \sigma' \end{array}$$

commutes. Let $\sigma_1' = (a_1, \dots, a_j)$, $\sigma_2^* = (a_{j+1}, \dots, a_m)$, $\sigma^* = (a_1', \dots, a_m')^{\mathrm{ord}}$, $a_{m+1} := 0$. Furthermore, let $b_1, \dots, b_{m+1}$ be integers such that $\sigma_1'\sigma_2' = (b_1, \dots, b_{m+1})^{\mathrm{ord}}$ where $a_i = b_i$ except $b_p = a_p - 1$ and possibly $b_q = a_q + 1$ for $a_p > a_q + 1$, $j \leq p < q \leq m+1$. We put $\sigma' = (b_1', \dots, b_{m+1}')^{\mathrm{ord}}$ and define b_i' in the following case distinction.

First assume $b_p > 0$ or $a_p = a_p'$. We put $b_i' = b_i - a_i + a_i'$. It follows that σ' can be obtained from σ^* by one elementary flattening, thus σ' dominates σ^*.

Now assume that $0 = b_p = a_p - 1 = a_p'$. It follows that no entry a_q is incremented, since otherwise we would have $a_q < 0$. By assumption, σ_2' is not empty, hence we can pick some $t \in \{j+1, \dots, m\} \setminus \{p\}$ with $b_t > 0$. If $a_t' = a_t - 1$, then we put $b_p' = b_p$ and $b_i' = b_i - a_i + a_i'$ for $i \neq p$; $\sigma' = \sigma^*$ follows (observe that $b_t' = b_t - 1$). Otherwise, if $a_t' = a_t$, then we put $b_p' = b_p$, $b_t' = b_t - 1$, and $b_i' = b_i - a_i + a_i'$ for $i \notin \{p, t\}$; in this case σ' arises from σ^* by an elementary flattening which decrements a_t'. It follows that σ' dominates σ^* in any case, hence in turn, σ' dominates σ as claimed. □

Repeated application of Lemma 4 yields the following result.

Corollary 1 *Let Γ and Γ' be sets of stairways such that every element of Γ is dominated by some element of Γ'. If $\Gamma \vdash_{N(s)} \sigma$ then σ is dominated by some σ' such that $\Gamma' \vdash_{N(s)} \sigma'$. In particular, $\Gamma \vdash_{N(s)} \varepsilon$ implies $\Gamma' \vdash_{N(s)} \varepsilon$.*

It would be interesting to know if there exists a more general notion of domination for which Corollary 1 holds.

Now it is easy to see that f_1 is computable: Assume that we want to decide whether $f_1(k) \leq s$. First decide whether $f_1(k-1) \leq s$ (we can inductively assume that this is possible); if $f_1(k-1) > s$

then clearly $f_1(k) > s$ and we are done. Otherwise, if $f_1(k-1) \leq s$, let T be an $N(s)$-derivation of ε from $(k-1)$, and let n denote the number of leaves of T. By changing all axioms of T from $(k-1)$ to (k), and by propagating this modification downward in T, we obtain an $N(s)$-derivation of the sequence 1^n, a sequence consisting of n 1s. Since every sequence of length at least n is dominated by 1^n, we can ignore all sequences of length greater than n in the saturation algorithm. On the other hand, all sequences containing an entry which is greater than k are dominated by (k); hence it follows that there is a finite number ($\leq (k+1)^n$) of sequences that have to be considered by the saturation algorithm. Hence it can be decided whether $f_1(k) \leq s$; thus f_1 is computable.

Theorem 3 *The function f_1 is computable.*

5 Restricting the Search Space

In this section we present further results which allow to speed up the computation of f_1.

5.1 A Deterministic Rule of Inference

Let $\sigma_1 = (a_1, \dots, a_j)$, $\sigma_2 = (a_{j+1}, \dots, a_n)$ be nonempty stairways, and let $(a_2, \dots, a_j, a_{j+2}, \dots, a_n)^{\text{ord}} = (b_1, \dots, b_{n-2})$. For given $s \geq 2$, we put $s' = \min(s, n) - 2$ and we define a stairway

$$\sigma_1 \oplus_s \sigma_2 := (a_1 - 1, a_j - 1, b_1 - 1, \dots, b_{s'} - 1, b_{s'+1}, \dots, b_{n-2})^{\text{ord}}$$

Thus, $\sigma_1 \oplus_s \sigma_2$ arises from $\sigma_1\sigma_2$ by decrementing the s largest entries of $\sigma_1\sigma_2$, ensuring that at least one entry of σ_1 and at least one entry of σ_2 is decremented.

Lemma 5 *Let σ_1, σ_2 be stairways. Then $\sigma_1 \oplus_s \sigma_2$ can be inferred from σ_1 and σ_2 by the rule $N(s)$; moreover, $\sigma_1 \oplus_s \sigma_2$ dominates all other sequences which can be inferred from σ_1 and σ_2 by the rule $N(s)$.*

Thus obtaining $\sigma_1 \oplus_s \sigma_2$ from σ_1, σ_2 is a special case of an inference by the rule $N(s)$. We denote the corresponding restricted form of the rule by $D(s)$.

Since every stairway is dominated by the empty sequence ε, Lemmas 4 and 5 immediately yield the following result.

Theorem 4 $f_1(k) = \min\{\, s : (k) \vdash_{D(s)} \varepsilon \,\} - 1$.

See Fig. 3 for a $D(8)$-derivation of ε from (5), displayed as a sequence of inference steps.

$\sigma_0 = (5)$
$\sigma_1 = \sigma_0 \oplus_8 \sigma_0 = (4,4)$
$\sigma_2 = \sigma_0 \oplus_8 \sigma_1 = (4,3,3)$
$\sigma_3 = \sigma_0 \oplus_8 \sigma_2 = (4,3,2,2)$
$\sigma_4 = \sigma_0 \oplus_8 \sigma_3 = (4,3,2,1,1)$
$\sigma_5 = \sigma_0 \oplus_8 \sigma_4 = (4,3,2,1)$
$\sigma_6 = \sigma_5 \oplus_8 \sigma_5 = (3,3,2,2,1,1)$
$\sigma_7 = \sigma_5 \oplus_8 \sigma_6 = (3,2,2,2,1,1,1,1)$
$\sigma_8 = \sigma_6 \oplus_8 \sigma_0 = (4,2,2,1,1)$
$\sigma_9 = \sigma_7 \oplus_8 \sigma_0 = (4,2,1,1,1,1,1)$
$\sigma_{10} = \sigma_8 \oplus_8 \sigma_0 = (4,3,1,1)$
$\sigma_{11} = \sigma_8 \oplus_8 \sigma_{10} = (3,3,2,1,1,1)$
$\sigma_{12} = \sigma_9 \oplus_8 \sigma_0 = (4,3,1)$
$\sigma_{13} = \sigma_{11} \oplus_8 \sigma_0 = (4,2,2,1)$
$\sigma_{14} = \sigma_{12} \oplus_8 \sigma_{13} = (3,3,2,1,1)$
$\sigma_{15} = \sigma_{12} \oplus_8 \sigma_{14} = (3,2,2,2,1)$
$\sigma_{16} = \sigma_{12} \oplus_8 \sigma_{15} = (3,2,2,1,1,1)$
$\sigma_{17} = \sigma_{16} \oplus_8 \sigma_0 = (4,2,1,1)$
$\sigma_{18} = \sigma_{17} \oplus_8 \sigma_{17} = (3,3,1,1)$
$\sigma_{19} = \sigma_{17} \oplus_8 \sigma_{18} = (3,2,2,1)$
$\sigma_{20} = \sigma_{17} \oplus_8 \sigma_{19} = (3,2,1,1,1)$
$\sigma_{21} = \sigma_{20} \oplus_8 \sigma_0 = (4,2,1)$
$\sigma_{22} = \sigma_{20} \oplus_8 \sigma_{21} = (3,2,1,1)$
$\sigma_{23} = \sigma_{20} \oplus_8 \sigma_{22} = (2,2,1,1,1)$
$\sigma_{24} = \sigma_{20} \oplus_8 \sigma_{23} = (2,1,1,1,1,1)$
$\sigma_{25} = \sigma_{24} \oplus_8 \sigma_0 = (4,1)$
$\sigma_{26} = \sigma_{24} \oplus_8 \sigma_{25} = (3,1)$
$\sigma_{27} = \sigma_{24} \oplus_8 \sigma_{26} = (2,1)$
$\sigma_{28} = \sigma_{24} \oplus_8 \sigma_{27} = (1,1)$
$\sigma_{29} = \sigma_{24} \oplus_8 \sigma_{28} = (1)$
$\sigma_{30} = \sigma_{29} \oplus_8 \sigma_{29} = \varepsilon$

Fig. 3. $D(8)$-derivation, certifying that $f(5) \leq f_1(5) \leq 7$.

5.2 Sequences of Length $s - 1$ are Enough

In the above argument for showing that f_1 is computable (Theorem 3) we established an upper bound for the maximum length of sequences we have to consider for deciding whether $f_1(k) \leq s$. This upper bound is very large and is not of practical help for actually determining $f_1(k)$ for small k. Next we present a construction which allows us to restrict the length of the sequences we have to consider to $s - 1$.

Let $s \geq 1$ and let $\sigma = (a_1, \ldots, a_n)$ be a stairway of length $n \geq s$. Consider the stairway

$$\sigma' = (a_1, \ldots, a_{s-2}, a_{s-1} + 1, a_s - 1, a_{s+1}, \ldots, a_n)^{\mathrm{ord}};$$

we say that σ' is obtained from σ by *elementary s-sloping*. We can apply s-sloping to σ repeatedly, until we end up with a sequence of length $s - 1$; we denote this sequence by $\sigma|^s$, and for any stairway σ of length $< s$, we put $\sigma|^s = \sigma$.

The next result allows us for the saturation algorithm to apply s-sloping before we add a new sequence to the cumulating set.

Theorem 5 *Let Γ be a set of stairways and let $\Gamma' := \{ \sigma|^s : \sigma \in \Gamma \}$. Then $\Gamma \vdash_{D(s)} \varepsilon$ if and only if $\Gamma' \vdash_{D(s)} \varepsilon$.*

Proof. ($\Leftarrow$) Since σ always dominates $\sigma|^t$, this direction of the theorem follows directly from Corollary 1.

($\Rightarrow$) Consider a $D(s)$-derivation T of ε from Γ. For every leaf v of T we count the number $k(v)$ of times we have to apply s-sloping to the sequence σ_v labeling v to obtain $\sigma_v|^s$. Let $k(T)$ denote the sum of $k(v)$ over all leaves of T. If $k(T) = 0$ then T is already a $D(s)$-derivation of ε from Γ', and we are done. Hence assume $k(T) > 0$. Below we describe a construction which modifies T in such a way that $k(T)$ is decreased; a repeated application of the construction yields to the case $k(T) = 0$.

We pick a leaf v_0 of T which is is labeled by $\sigma_0 = (a_1, \ldots, a_n)$ for $n \geq s$.

Let $v_0, \ldots, v_r$ be the sequence of vertices on the path P from v_0 to the root v_r of T. We introduce now a notion which will allow us to talk precisely about what happens to the entries of σ_0 on the path P.

Consider an entry a_j of σ_0. Following the path P from v_0 to v_r, we can track the entry a_j. At each step of inference, it is either decremented or it retains its value, until its value reaches 0 (we can always find its new position after sorting the sequence). We use this procedure to track $a_1, \ldots, a_n$ so that at v_i their values are represented by the sequence $A_i := (a_1^{(i)}, \ldots, a_n^{(i)})$, $i = 0, \ldots, r$. Using the freedom in the choice of A_i, we can make sure that

$$a_1^{(i)} \geq \cdots \geq a_{s-1}^{(i)} \quad \text{for } i = 0, \ldots, r. \tag{4}$$

We call $\tau = (a_1^{(i)}, \ldots, a_n^{(i)})_{i=0}^{r}$ a *trace* of v_0. Note that in general, v_0 has several possible traces. Since T is a $D(s)$-derivation, it follows that for any transition from A^i to A^{i+1}, if an entry of A^i is decremented, all strictly larger elements of A^i are decremented as well; we refer to this property of the trace as *$>$-preference*. For entries of A^i of equal value, we have some freedom in the choice of the trace. We assume that if an entry $a_t^{(i)}$ is decremented for $t \geq s$, then all entries $a_{t'}^{(i)} = a_t^{(i)}$ for $t' < s$ are decremented as well. We refer to this property of the trace as *$=$-preference*.

Let $i_0 \in \{1, \ldots, r-1\}$ be the smallest index such that $a_s^{(i_0+1)} = a_s^{(i_0)} - 1$ (such i_0 exists, since the root v_r is labeled by the empty sequence, and so $A_r = (0, \ldots, 0)$). At the transition from A_{i_0} to A_{i_0+1} at most $s-1$ entries are decremented; by the pigeon hole principle it follows that at least one $a_t^{(i_0)}$, $t < s$, is not decremented. $<$-preference implies $a_t^{(i_0)} \leq a_s^{(i_0)}$, and $=$-preference implies $a_t^{(i_0)} < a_s^{(i_0)}$. In view of (4), we may assume that $t = s - 1$, therefore $a_{s-1}^{(i_0)} < a_s^{(i_0)}$.

Now we modify the labels of the vertices v_i, $i = 0, \ldots, i_0$, as follows. We can replace in σ_{v_i} the entries $a_{s-1}^{(i)}$ and $a_s^{(i)}$ by $a_{s-1}^{(i)} + 1$ and $a_s^{(i)} - 1$, respectively (by assumption, $a_s^{(i)} = a_s$ for $i \leq i_0$). Let T' denote the new labeled tree. To show that T' is an $N(s)$-derivation, it suffices to justify the labels of $v_0, \ldots, v_{i_0+1}$ by the rule $N(s)$. This is easy for $v_0, \ldots, v_{i_0}$. By assumption, the inference that yields the label v_{i_0+1} involves decrementing $a_s^{(i_0)}$, ($a_s^{(i_0+1)} = a_s^{(i_0)} - 1$), but $a_{s-1}^{(i_0)}$ is not changed

$(a_{s-1}^{(i_0+1)} = a_{s-1}^{(i_0)})$. In T', we simply swap the roles of these two entries, and obtain the original label of v_{i_0+1}. Hence T' is indeed an $N(s)$-derivation and, as we have applied elementary s-sloping to the label of v_0, $k(T') = k(T) - 1$.

In order to complete our inductive argument, we transform the $N(s)$-derivation T' into a $D(s)$-derivation T'' such that $k(T'') \leq k(T')$. We apply Lemmas 4 and 5 along the path P. That is, assume that vertex v_i, $1 \leq i \leq r$ is labeled by a sequence λ, and that its parents v_{i-1} and v'_{i-1} are labeled by λ_1 and λ_2, respectively. If we change λ_1 to some sequence λ'_1 which dominates λ_1, then, in view of Lemmas 4 and 5, we can change λ to $\lambda' := \lambda'_1 \oplus_s \lambda_2$ (λ' dominates λ). We apply this re-labeling to $v_1, v_2, \ldots$ until we reach a vertex $v_{r'}$ which receives the label ε. The subtree T'' rooted in $v_{r'}$ is now a $D(s)$-derivation with $k(T'') \leq k(T') < k(T)$ as claimed. Hence, by iteration, we are finally left with a $D(s)$-derivation T^* with $k(T^*) = 0$, which is a $D(s)$-derivation of ε from Γ'. This completes the proof of the theorem. □

Appendix: A $D(12)$-Derivation, Certifying that $f(6) \leq f_1(6) \leq 11$

$\sigma_0 = (6)$
$\sigma_1 = \sigma_0 \oplus_{12} \sigma_0$
$\sigma_2 = \sigma_0 \oplus_{12} \sigma_1$
$\sigma_3 = \sigma_0 \oplus_{12} \sigma_2$
$\sigma_4 = \sigma_0 \oplus_{12} \sigma_3$
$\sigma_5 = \sigma_0 \oplus_{12} \sigma_4$
$\sigma_6 = \sigma_0 \oplus_{12} \sigma_5$
$\sigma_7 = \sigma_1 \oplus_{12} \sigma_1$
$\sigma_8 = \sigma_1 \oplus_{12} \sigma_6$
$\sigma_9 = \sigma_1 \oplus_{12} \sigma_8$
$\sigma_{10} = \sigma_1 \oplus_{12} \sigma_9$
$\sigma_{11} = \sigma_1 \oplus_{12} \sigma_{10}$
$\sigma_{12} = \sigma_1 \oplus_{12} \sigma_{11}$
$\sigma_{13} = \sigma_2 \oplus_{12} \sigma_{12}$
$\sigma_{14} = \sigma_6 \oplus_{12} \sigma_{12}$
$\sigma_{15} = \sigma_6 \oplus_{12} \sigma_{13}$
$\sigma_{16} = \sigma_7 \oplus_{12} \sigma_{12}$
$\sigma_{17} = \sigma_7 \oplus_{12} \sigma_{13}$
$\sigma_{18} = \sigma_{14} \oplus_{12} \sigma_0$
$\sigma_{19} = \sigma_{15} \oplus_{12} \sigma_0$
$\sigma_{20} = \sigma_{16} \oplus_{12} \sigma_0$
$\sigma_{21} = \sigma_{17} \oplus_{12} \sigma_0$
$\sigma_{22} = \sigma_{18} \oplus_{12} \sigma_0$
$\sigma_{23} = \sigma_{18} \oplus_{12} \sigma_1$
$\sigma_{24} = \sigma_{18} \oplus_{12} \sigma_{22}$
$\sigma_{25} = \sigma_{19} \oplus_{12} \sigma_0$
$\sigma_{26} = \sigma_{20} \oplus_{12} \sigma_1$
$\sigma_{27} = \sigma_{21} \oplus_{12} \sigma_0$
$\sigma_{28} = \sigma_{23} \oplus_{12} \sigma_{25}$
$\sigma_{29} = \sigma_{23} \oplus_{12} \sigma_{28}$
$\sigma_{30} = \sigma_{24} \oplus_{12} \sigma_0$
$\sigma_{31} = \sigma_{25} \oplus_{12} \sigma_{27}$
$\sigma_{32} = \sigma_{25} \oplus_{12} \sigma_{28}$
$\sigma_{33} = \sigma_{25} \oplus_{12} \sigma_{32}$
$\sigma_{34} = \sigma_{26} \oplus_{12} \sigma_{31}$
$\sigma_{35} = \sigma_{27} \oplus_{12} \sigma_{31}$
$\sigma_{36} = \sigma_{29} \oplus_{12} \sigma_0$
$\sigma_{37} = \sigma_{30} \oplus_{12} \sigma_{34}$
$\sigma_{38} = \sigma_{33} \oplus_{12} \sigma_0$
$\sigma_{39} = \sigma_{35} \oplus_{12} \sigma_{35}$
$\sigma_{40} = \sigma_{36} \oplus_{12} \sigma_0$
$\sigma_{41} = \sigma_{37} \oplus_{12} \sigma_0$
$\sigma_{42} = \sigma_{38} \oplus_{12} \sigma_{38}$
$\sigma_{43} = \sigma_{38} \oplus_{12} \sigma_{40}$
$\sigma_{44} = \sigma_{38} \oplus_{12} \sigma_{42}$
$\sigma_{45} = \sigma_{39} \oplus_{12} \sigma_0$
$\sigma_{46} = \sigma_{40} \oplus_{12} \sigma_{40}$
$\sigma_{47} = \sigma_{40} \oplus_{12} \sigma_{43}$
$\sigma_{48} = \sigma_{41} \oplus_{12} \sigma_0$
$\sigma_{49} = \sigma_{42} \oplus_{12} \sigma_{46}$
$\sigma_{50} = \sigma_{42} \oplus_{12} \sigma_{47}$
$\sigma_{51} = \sigma_{42} \oplus_{12} \sigma_{48}$
$\sigma_{52} = \sigma_{44} \oplus_{12} \sigma_0$
$\sigma_{53} = \sigma_{45} \oplus_{12} \sigma_0$
$\sigma_{54} = \sigma_{49} \oplus_{12} \sigma_{52}$
$\sigma_{55} = \sigma_{50} \oplus_{12} \sigma_0$
$\sigma_{56} = \sigma_{51} \oplus_{12} \sigma_{53}$
$\sigma_{57} = \sigma_{51} \oplus_{12} \sigma_{56}$
$\sigma_{58} = \sigma_{52} \oplus_{12} \sigma_{55}$
$\sigma_{59} = \sigma_{52} \oplus_{12} \sigma_{58}$
$\sigma_{60} = \sigma_{53} \oplus_{12} \sigma_{58}$
$\sigma_{61} = \sigma_{53} \oplus_{12} \sigma_{59}$
$\sigma_{62} = \sigma_{54} \oplus_{12} \sigma_0$
$\sigma_{63} = \sigma_{55} \oplus_{12} \sigma_{55}$
$\sigma_{64} = \sigma_{55} \oplus_{12} \sigma_{58}$
$\sigma_{65} = \sigma_{57} \oplus_{12} \sigma_0$
$\sigma_{66} = \sigma_{58} \oplus_{12} \sigma_{60}$
$\sigma_{67} = \sigma_{60} \oplus_{12} \sigma_{62}$
$\sigma_{68} = \sigma_{60} \oplus_{12} \sigma_{66}$
$\sigma_{69} = \sigma_{61} \oplus_{12} \sigma_{67}$
$\sigma_{70} = \sigma_{63} \oplus_{12} \sigma_0$
$\sigma_{71} = \sigma_{64} \oplus_{12} \sigma_0$
$\sigma_{72} = \sigma_{65} \oplus_{12} \sigma_{71}$
$\sigma_{73} = \sigma_{66} \oplus_{12} \sigma_{71}$
$\sigma_{74} = \sigma_{68} \oplus_{12} \sigma_0$
$\sigma_{75} = \sigma_{69} \oplus_{12} \sigma_0$
$\sigma_{76} = \sigma_{70} \oplus_{12} \sigma_{74}$
$\sigma_{77} = \sigma_{72} \oplus_{12} \sigma_{76}$
$\sigma_{78} = \sigma_{73} \oplus_{12} \sigma_0$
$\sigma_{79} = \sigma_{75} \oplus_{12} \sigma_{75}$
$\sigma_{80} = \sigma_{75} \oplus_{12} \sigma_{79}$
$\sigma_{81} = \sigma_{75} \oplus_{12} \sigma_{80}$
$\sigma_{82} = \sigma_{75} \oplus_{12} \sigma_{81}$
$\sigma_{83} = \sigma_{77} \oplus_{12} \sigma_{80}$
$\sigma_{84} = \sigma_{78} \oplus_{12} \sigma_{83}$
$\sigma_{85} = \sigma_{79} \oplus_{12} \sigma_{79}$
$\sigma_{86} = \sigma_{79} \oplus_{12} \sigma_{82}$
$\sigma_{87} = \sigma_{79} \oplus_{12} \sigma_{86}$
$\sigma_{88} = \sigma_{79} \oplus_{12} \sigma_{87}$
$\sigma_{89} = \sigma_{80} \oplus_{12} \sigma_{88}$
$\sigma_{90} = \sigma_{80} \oplus_{12} \sigma_{89}$
$\sigma_{91} = \sigma_{84} \oplus_{12} \sigma_0$
$\sigma_{92} = \sigma_{85} \oplus_{12} \sigma_{90}$
$\sigma_{93} = \sigma_{85} \oplus_{12} \sigma_{92}$
$\sigma_{94} = \sigma_{91} \oplus_{12} \sigma_{93}$
$\sigma_{95} = \sigma_{91} \oplus_{12} \sigma_{94}$
$\sigma_{96} = \sigma_{93} \oplus_{12} \sigma_{95}$
$\sigma_{97} = \sigma_{93} \oplus_{12} \sigma_{96}$
$\sigma_{98} = \sigma_{97} \oplus_{12} \sigma_0$
$\sigma_{99} = \sigma_{97} \oplus_{12} \sigma_{98}$
$\sigma_{100} = \sigma_{97} \oplus_{12} \sigma_{99}$
$\sigma_{101} = \sigma_{97} \oplus_{12} \sigma_{100}$
$\sigma_{102} = \sigma_{97} \oplus_{12} \sigma_{101}$
$\sigma_{103} = \sigma_{102} \oplus_{12} \sigma_{102} = \varepsilon$

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