# Looking Algebraically at Tractable Quantified Boolean Formulas

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**Abstract.** We make use of the algebraic theory that has been used to study the complexity of constraint satisfaction problems, to investigate tractable quantified boolean formulas. We present a pair of results: the first is a new and simple algebraic proof of the tractability of quantified 2-satisfiability; the second is a purely algebraic characterization of models for quantified Horn formulas that were given by Büning, Subramani, and Zhao, and described proof-theoretically.

## 1 Introduction

An instance of the *generalized satisfiability* problem is a set of constraints, where a constraint is a relation over the two-element domain $\{0, 1\}$ paired with a variable tuple having the same arity as the relation; the question is to decide whether or not there is a $0 - 1$ assignment to all of the variables satisfying all of the constraints. A constraint is satisfied under an assignment if the variable tuple mapped under the assignment falls into the corresponding relation. Schaefer was the first to consider the generalized satisfiability problem [22]. He proved a now famous complexity classification theorem, showing that for any constraint language–a set of relations that can be used to express constraints–the generalized satisfiability problem over that constraint language is either in P or is NP-complete. This result has spawned a number of extensions and generalizations (see for example [14]). Analogous classification theorems have been proven for variants of the satisfiability problem such as quantified satisfiability [14]; also, there has been much recent work on establishing a complexity classification theorem for the general *constraint satisfaction problem (CSP)*, in which relations over domains of size greater than two are permitted.

In the nineties, an *algebraic viewpoint* on constraints was established that made it possible to approach the task of performing CSP complexity classification using tools from universal algebra [20, 18]; this viewpoint has produced a rich line of results, including [19, 15, 7, 4–6, 8, 3]. One fruit of this viewpoint has been a perspective on the tractable cases of satisfiability established by Schaefer's theorem, which include *2-satisfiability* and *Horn satisfiability*. For instance, an exact algebraic characterization of CSPs where it is possible to "go from local to global consistency" has been given [19]; the 2-satisfiability problem yields the simplest non-trivial example from this class of CSPs.

In this paper, we demonstrate that the algebraic viewpoint that has been used to study the complexity of generalized satisfiability and the CSP can be used to derive new and interesting results concerning the quantified satisfiability problem–despite the fact that the complexity classification program for which the algebraic viewpoint was originally developed, has been completed in the two-element case (for both standard satisfiability [22] and quantified satisfiability [14]). It is our hope that this paper will stimulate further work on satisfiability that utilizes this algebraic viewpoint.

We present two results, one on quantified 2-satisfiability and the other on quantified Horn satisfiability; these two particular cases of the quantified satisfiability problem are known to be tractable [1, 21, 10]. Our results are as follows.

First, we give a new algebraic proof that quantified 2-satisfiability is tractable in polynomial time. In particular, we analyze an algorithm of Gent and Rowley [17]. From an implementation standpoint, this algorithm is extremely simple (and in the spirit of an algorithm for 2-satisfiability given by Del Val [16]): other than simple manipulations such as setting and removing variables, the only conceptual primitive used is unit resolution. We establish the correctness of this algorithm via a relatively simple and succinct proof which, unlike the proof of correctness given in [17], does not rely on the theory developed in [1]. In fact, we establish a more widely applicable result: we give a generalization of the algorithm given in [17] and demonstrate that it yields a general, algebraic sufficient condition for the tractability of the quantified constraint satisfaction problem. Our presentation of this new tractability result is self-contained, though the result was inspired by ideas in [13].

Second, we give a purely algebraic characterization of models for quantified boolean Horn formulas that were identified by Büning, Subramani, and Zhao [11]. They demonstrated that any true quantified Horn formula has a model of a particularly simple form–where every existentially quantified variable is set to either a constant or a conjunction of universally quantified variables. For any true quantified Horn formula $\Phi$, they identified a model of this form described using the clauses derivable from $\Phi$ in Q-unit-resolution, a proof system known to be sound and complete for quantified Horn formulas [10]. We give an equivalent description of the models that they identified by making use of the semilattice structure possessed by the models of a quantified Horn formula, along with some natural homomorphisms among quantified Horn formulas that we introduce.

## 2 Tractability of Quantified 2-Satisfiability

The following is the basic terminology of quantified constraint satisfaction that we will use in this section. A *constraint* is an expression $R(v_1, \ldots, v_k)$ where each $v_i$ is a variable, and $R \subseteq B^k$ is an arity $k$ relation over a finite domain $B$. The constraint $R(v_1, \ldots, v_k)$ is true under an interpretation $f : V \to B$ defined on the variables $v_i$ if $(f(v_1), \ldots, f(v_k)) \in R$. A quantified formula (over domain $B$) is an expression of the form $Q_1 v_1 \ldots Q_n v_n \phi$ where each $Q_i$ is a quantifier from the set $\{\forall, \exists\}$ and where $\phi$ is a conjunction of constraints over the variables $\{v_1, \ldots, v_n\}$. A constraint language is defined to be a set of relations over the same domain. The quantified constraint satisfaction problem over a constraint language $\Gamma$, denoted by $\mathsf{QCSP}(\Gamma)$, is the problem of deciding, given as input a quantified formula $\Phi$ having relations from $\Gamma$, whether or not $\Phi$ is true.

We now review some of the key elements of the algebraic viewpoint that has been fruitful in the study of constraint satisfaction [20, 18]. The central notion of this viewpoint is the concept of *polymorphism*. Let $f : B^m \to B$ be an $m$-ary operation on $B$, let $R$ be a relation over $B$, and let $k$ denote the arity of $R$. We say that $f$ is a *polymorphism* of $R$, or that $R$ is *invariant* under $f$, if for all (not necessarily distinct) tuples $(b_1^1, \ldots, b_k^1), \ldots, (b_1^m, \ldots, b_k^m)$ in $R$, the tuple

$$(f(b_1^1, \ldots, b_1^m), \ldots, f(b_k^1, \ldots, b_k^m))$$

belongs also to $R$. For example, let $T$ be the binary boolean relation

$$\{(0,0), (0,1), (1,1)\}$$

having the property that the constraint $T(u, v)$ is equivalent to the clause $(\neg u \vee v)$, and let $\mathsf{maj} : \{0,1\}^3 \to \{0,1\}$ be the ternary majority function on $\{0,1\}$. It is not difficult to verify that $\mathsf{maj}$ is a polymorphism of $T$. In fact, it is known that a boolean relation $R$ is invariant under $\mathsf{maj}$ if and only if any constraint over $R$ is equivalent to a 2-satisfiability formula.

In general, there is an intimate relationship between polymorphisms and conjunctive formulas: it has been shown that the complexity of $\mathsf{QCSP}(\Gamma)$ is determined by the set of all polymorphisms common to all relations in $\Gamma$, denoted by $\mathsf{Pol}(\Gamma)$. More precisely, if $\Gamma_1$ and $\Gamma_2$ are finite constraint languages such that $\mathsf{Pol}(\Gamma_1)$ and $\mathsf{Pol}(\Gamma_2)$ contain the same functions, then $\mathsf{QCSP}(\Gamma_1)$ and $\mathsf{QCSP}(\Gamma_2)$ are reducible to each other via polynomial-time many-one reductions [2]. Furthermore, the literature contains many results that link the complexity of $\mathsf{CSP}(\Gamma)$ and $\mathsf{QCSP}(\Gamma)$ with the presence (or absence) of functions of certain types in $\mathsf{Pol}(\Gamma)$.[3] As an example, it is known that the presence

[3] By $\mathsf{CSP}(\Gamma)$, we denote the standard CSP over $\Gamma$–that is, the restriction of $\mathsf{QCSP}(\Gamma)$ to formulas having only existential quantifiers.

in $\mathsf{Pol}(\Gamma)$ of the ternary *dual discriminator* function $t : B^3 \to B$ defined by

$$t(x, y, z) = \begin{cases} x \text{ if } x = y \\ z \text{ otherwise} \end{cases}$$

implies that $\mathsf{QCSP}(\Gamma)$ is solvable in polynomial time [2]. This implies the tractability of quantified 2-satisfiability, since the dual discriminator function over a two-element domain $\{0, 1\}$ is equivalent to the $\mathsf{maj}$ function; however, the proof given in [2] is fairly involved.

In this section, we will use an algebraic approach slightly different from the polymorphism-based approach. Our approach is based on a new notion called the *extended set polymorphism*, which we have studied also in the context of CSP complexity [13]. Let $R$ be a relation, say $k$-ary, over a domain $B$. An extended set function $f$ is any function with domain $\mathcal{P}(B) \times B$ and range $B$. (Here, we use $\mathcal{P}(B)$ to denote the set containing all non-empty subsets of $B$, that is, the power set of $B$ excluding the empty set.) We say that an extended set function $f$ is an *extended set polymorphism* of $R$ (or, $R$ is *invariant* under $f$) if for every $m \geq 1$ and all tuples $(b_1^1, \ldots, b_k^1), \ldots, (b_1^m, \ldots, b_k^m), (c_1, \ldots, c_k)$ in $R$, the tuple

$$(f(\{b_1^1, \ldots, b_1^m\}, c_1), \ldots, f(\{b_k^1, \ldots, b_k^m\}, c_k))$$

belongs also to $R$. As an example, consider again the boolean relation $T$ as defined above, and let $g : \mathcal{P}(\{0, 1\} \times \{0, 1\} \to \{0, 1\}$ be defined as

$$g(S, b) = \begin{cases} s \text{ if } |S| = 1 \text{ and } S = \{s\} \\ b \text{ otherwise} \end{cases}$$

It is immediate to verify that $g$ is an extended set polymorphism of $T$. Indeed, with a little bit of effort it can be proven that a relation $R$ is invariant under $g$ if and only if any constraint over $R$ is equivalent to a 2-satisfiability formula.

As with regular polymorphisms, we will say that a constraint language $\Gamma$ is invariant under an extended set function $f$ if every relation in $\Gamma$ is invariant under $f$. We have the following general tractability result.

**Theorem 1.** *Let $\Gamma$ be a constraint language over domain $B$ invariant under an extended set function $f : \mathcal{P}(B) \times B \to B$ such that $f(B, b) = b = f(\{b\}, c)$ for all $b, c \in B$. Then, $\mathsf{QCSP}(\Gamma)$ is solvable in polynomial time.*

We can derive the tractability of quantified 2-satisfiability from Theorem 1 and the fact that the set of satisfying assignments of a 2-clause is invariant under the extended set function $g$ described above. In fact, we can derive from Theorem 1 the tractability of any constraint language $\Gamma$ invariant under the dual discriminator function $t : B^3 \to B$ described above.

The algorithm used to establish Theorem 1 makes use of the notion of arc consistency. We say that a conjunction of constraints $\phi$ is *arc consistent* if when $R(w_1, \ldots, w_k)$, $R'(w'_1, \ldots, w'_{k'})$ are two constraints in $\phi$ such that $w_i = w'_j$, then the projection of $R$ onto the $i$th coordinate is equal to the projection of $R'$ onto the $j$th coordinate. This common projection is called the *domain* of the variable $w_i = w'_j$. We say that *arc consistency can be established* on a conjunction of constraints if there is an equivalent conjunction of constraints that is arc consistent and where no variable has empty domain. It is well-known that testing to see if arc consistency can be established is performable in polynomial time.

The algorithm (for Theorem 1), which generalizes the algorithm for quantified 2-satisfiability given in [17], is as follows. In each step of the algorithm, the outermost quantified variable is eliminated. Let $\Phi = Q_1 v_1 \ldots Q_n v_n \phi$ be a quantified formula. First suppose that $Q_1$ is an existential quantifier. In this case, the algorithm attempts to find a value $b$ in the domain $B$ such that arc consistency can be established on $\phi[v_1 = b]$ where every universal variable has a *full* domain, that is, domain equal to $B$. If there is no such value, the formula $\Phi$ is false. Otherwise, the algorithm sets $v_1$ to such a value, and continues. Next, suppose that $Q_1$ is a universal quantifier. In this case, the algorithm attempts to ensure that for *every* value $b$ in the domain $B$, arc consistency can be established on $\phi[v_1 = b]$ where every universal variable (other than $v_1$) has a full domain. If there is any such value where this is not the case, the formula is false. Otherwise, the algorithm sets $v_1$ to *any* value, and continues.

To prove the correctness of this algorithm (for $\mathsf{QCSP}(\Gamma)$ satisfying the hypothesis of Theorem 1), we use the following characterization of true quantified formulas: a quantified formula $\Phi = Q_1 v_1 \dots Q_n v_n \phi$ is true if there is a set $\mathcal{S}$ of satisfying assignments for $\phi$ satisfying the two following properties:

(a) For every partial assignment $\alpha$ to the universal variables, there exists an extension $\beta$ of $\alpha$ in $\mathcal{S}$.
(b) Suppose that $\alpha, \beta \in \mathcal{S}$ are two assignments such that for every universal variable $y$ preceding an existential variable $x$, we have that $\alpha(y) = \beta(y)$. Then, it holds that $\alpha(x) = \beta(x)$.

*Proof.* We establish the correctness of the algorithm by proving that whenever a variable is eliminated, truth of the formula is preserved. To demonstrate this, it suffices to prove the following fact: for any $b, b' \in B$, if arc consistency can be established on $\phi[v_1 = b]$ where every universal variable (coming after $v_1$) has full domain, and $\Phi[v_1 = b']$ is true, then $\Phi[v_1 = b]$ is true. Let $\mathcal{S}'$ be a set of satisfying assignments for $\phi[v_1 = b']$ with the above two properties. Let $V$ denote the set of variables $\{v_1, \dots, v_n\}$. By definition of arc consistency, there exists a mapping $a : V \to \mathcal{P}(B)$ such that $a(v_1) = \{b\}$, $a(y) = B$ for all universal variables $y$, and for any constraint $R(w_1, \dots, w_k)$ in $\phi$, there are tuples $(b_1^1, \dots, b_k^1), \dots, (b_1^m, \dots, b_k^m) \in R$ where $a(w_i) = \{b_i^1, \dots, b_i^m\}$ for all $i = 1, \dots, k$. Let $\mathcal{S}$ be the set of assignments $h : V \setminus \{v_1\} \to B$ of the form $h(v) = f(a(v), h'(v))$ where $h' \in \mathcal{S}'$. It is straightforward to verify that $\mathcal{S}$ evidences the truth of $\Phi[v_1 = b]$. □

## 3 Quantified Horn Formulas and BSZ models

In this section, we give a purely algebraic characterization of the models for true quantified Horn formulas provided in [11], which we call BSZ models.

Before proceeding, we introduce some basic terminology and notation for this section. We will often, for sake of notation, restrict attention to quantified formulas of the form $\forall y_1 \exists x_1 \dots \forall y_n \exists x_n \phi$, that is, formulas where there is a strict alternation between universal and existential quantifiers. In our quantified formulas, $\phi$ will always denote a boolean formula in conjunctive normal form, that is, a conjunction of clauses. Recall that a conjunction of clauses $\phi$ is a Horn formula if all of its clauses contain at most one positive literal. We define an *existential unit clause* to be a clause $C$ such that there is exactly one existential literal $l \in C$, the literal $l$ is a postive literal, and all literals in $C \setminus \{l\}$ come before $l$ in the quantification order of the formula in which $C$ appears.

A *strategy* for a quantified formula $\Phi = \forall y_1 \exists x_1 \dots \forall y_n \exists x_n \phi$ is a sequence of mappings $\{\sigma_i : \{0,1\}^i \to \{0,1\}\}_{i \in [n]}$, where $[n]$ denotes the set containing the first $n$ positive integers, $\{1, \dots, n\}$. A strategy $\{\sigma_i : \{0,1\}^i \to \{0,1\}\}_{i \in [n]}$ is a *model* of $\Phi$ if for all $a_1, \dots, a_n \in \{0,1\}$, it holds that the assignment mapping $y_i$ to $a_i$ and $x_i$ to $\sigma_i(a_1, \dots, a_i)$ (for all $i \in [n]$) satisfies $\phi$. We consider a quantified formula $\Phi$ to be true if it has a model, and use $\mathcal{M}_\Phi$ to denote the set of all models of $\Phi$.

Before defining the BSZ model for a quantified Horn formula, we need to introduce the following proof system for quantified Horn formulas.

**Definition 2.** *[10] The* Q-unit-resolution *proof system is defined as follows:*
*Let $\Phi = Q_1 v_1 \dots Q_m v_m \phi$ be a quantified boolean formula.*

- *For any clause $C \in \phi$, $\Phi \vdash C$.*
- *If $\Phi \vdash C$, $\Phi \vdash C'$, $C$ is an existential unit clause with existential variable $x$, and $C'$ is a clause containing $\neg x$, then $\Phi \vdash (C \cup C') \setminus \{x, \neg x\}$.*
- *If $\Phi \vdash C$ and $l \in C$ is a literal over a universal variable coming after all other literals in $C$ in the quantification order, then $\Phi \vdash C \setminus \{l\}$.*

It has been shown that this proof system is sound and complete for the class of all quantified Horn formulas [10]: for any quantified Horn formula $\Phi$, the empty clause is derivable from $\Phi$ (that is, $\Phi \vdash \emptyset$) if and only if $\Phi$ is false. Having defined this proof system, we can now define the BSZ model for a quantified Horn formula $\Phi$, which is described in terms of the clauses derivable from $\Phi$.

**Definition 3.** *[11] The* BSZ model *of a true quantified Horn formula $\Phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n \phi$ is defined as follows. Let $U$ denote the set of all existential variables $x$ such that there exists an existential unit clause $C$ where $x \in C$ and $\Phi \vdash C$. For an existential variable $x$, let $W(x)$ denote the set containing all sets of negative universal literals $C$ such that $C \cup \{x\}$ is an existential unit clause derivable from $\Phi$ (using Q-unit-resolution); and, let $V(x)$ denote the set $\cap_{C \in W(x)} C$.*

*The BSZ model of $\Phi$ is the model $\{\sigma_i\}_{i\in[n]}$ such that*

$$\sigma_i(y_1,\ldots,y_i) = \begin{cases} 0 & \text{if } x_i \notin U \\ \wedge_{l \in V(x_i)} \neg l & \text{if } x_i \in U, V(x_i) \neq \emptyset \\ 1 & \text{if } x_i \in U, V(x_i) = \emptyset \end{cases}$$

In what follows, we give a purely algebraic characterization of the BSZ model of a true quantified Horn formula. We first observe that if for any two models $\Sigma = \{\sigma_i : \{0,1\}^i \to \{0,1\}\}_{i\in[n]}$, $\Sigma' = \{\sigma'_i : \{0,1\}^i \to \{0,1\}\}_{i\in[n]}$ of a quantified Horn formula $\Phi$, the strategy $\Sigma \wedge \Sigma'$ defined as $\{\sigma_i \wedge \sigma'_i\}_{i\in[n]}$ can be verified to also be a model for $\Phi$. This follows from the fact that the operation $\wedge$ is a polymorphism of any Horn clause: when $\phi$ is a Horn clause (or more generally, a conjunction of Horn clauses) over a variable set $V$, and $f : V \to \{0,1\}$ and $f' : V \to \{0,1\}$ are interpretations both satisfying $\phi$, the interpretation $f \wedge f'$ also satisfies $\phi$. Because the operation $\wedge$ (applied to strategies as above) is associative, commutative, and idempotent, we have the following observation.

**Proposition 4.** *For all quantified Horn formulas $\Phi$, $(\mathcal{M}_\Phi, \wedge)$ is a semilattice.*

For any quantified Horn formula $\Phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n \phi$, let $\Phi[y_k]$ be the formula obtained from $\Phi$ by removing all universal variables $y_i$ from the quantifier prefix, except for $y_k$, and instantiating all instances of variables $y_i$ (with $i \neq k$) in $\phi$ with the value 1. We define a mapping $\mathbf{d}_{y_k}$ from the set of strategies for $\Phi$ to the set of strategies for $\Phi[y_k]$ as follows. When $\Sigma = \{\sigma_i\}_{i\in[n]}$ is a strategy for $\Phi$, define $\mathbf{d}_{y_k}(\Sigma)$ to be the strategy $\Sigma' = \{\sigma'_i\}_{i\in[n]}$ for $\Phi[y_k]$ such that for $i < k$, it holds that $\sigma'_i = \sigma_i(1,\ldots,1)$, and for $i \geq k$, it holds that $\sigma'_i(y_k) = \sigma_i(1,\ldots,1,y_k,1,\ldots,1)$, where in the right hand side expression, the $k$th coordinate contains $y_k$, and all other coordinates contain 1. It is clear that if $\Sigma$ is a model of $\Phi$, then $\mathbf{d}_{y_k}(\Sigma)$ is a model of $\Phi[y_k]$. We define $\mathbf{d}$ to be the mapping taking a strategy $\Sigma$ of $\Phi$ to the $n$-tuple $(\mathbf{d}_{y_1}(\Sigma),\ldots,\mathbf{d}_{y_n}(\Sigma))$, and call $\mathbf{d}$ the deconstruction mapping, as it deconstructs a model for $\Phi$ into models for the various $\Phi[y_k]$.

Interestingly, given models for the various $\phi_{y_k}$, we can construct a model for $\phi$. Let $\mathbf{r}$ be the mapping taking an $n$-tuple of strategies $(\Sigma_1,\ldots,\Sigma_n)$, where $\Sigma_k = \{\sigma^k_i\}_{i\in[n]}$ is a strategy for $\phi_{y_k}$, to the strategy $\Sigma = \{\sigma_i : \{0,1\}^i \to \{0,1\}\}_{i\in[n]}$ for $\phi$, where for $i \in [n]$, the mapping $\sigma_i$ is defined by

$$\sigma_i(y_1,\ldots,y_i) = (\sigma^1_i(y_1) \wedge \ldots \wedge \sigma^i_i(y_i)) \wedge (\sigma^{i+1}_i \wedge \ldots \wedge \sigma^n_i).$$

We call $\mathbf{r}$ the *reconstruction* mapping. We have the following result concerning the deconstruction and reconstruction mappings.

**Theorem 5.** *For all quantified Horn formulas $\Phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n \phi$,*

- *the deconstruction mapping $\mathbf{d}$ is a homomorphism from the semilattice $(\mathcal{M}_\Phi, \wedge)$ to the semilattice $(\mathcal{M}_{\Phi[y_1]}, \wedge) \times \cdots \times (\mathcal{M}_{\Phi[y_n]}, \wedge)$; and,*
- *the reconstruction mapping $\mathbf{r}$ is a homomorphism from the semilattice $(\mathcal{M}_{\Phi[y_1]}, \wedge) \times \cdots \times (\mathcal{M}_{\Phi[y_n]}, \wedge)$ to the semilattice $(\mathcal{M}_\Phi, \wedge)$.*

From Theorem 5, we can deduce the following corollary: if for a quantified Horn formula $\Phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n \phi$ it holds that all of the formulas $\Phi[y_1],\ldots,\Phi[y_n]$ are true, then $\Phi$ itself is true. This fact was observed, for instance, in [10]; however, the proof of Theorem 5 gives a purely algebraic justification of it.

It is well-known that every semilattice $(S, \oplus)$ has a maximal element $m$ such that $m \oplus s = s \oplus m = m$ for all $s \in S$. For every quantified Horn formula $\Phi$, we let $\Upsilon_\Phi$ denote the maximal element of $(\mathcal{M}_\Phi, \wedge)$. Notice that for the pointwise ordering $\leq$ on functions where $1 \leq 0$, when $\Sigma = \{\sigma_i\}_{i\in[n]}$ is a model of $\Phi$ and $\{\upsilon_i\}_{i\in[n]}$ denotes $\Upsilon_\Phi$, it holds that $\sigma_i \leq \upsilon_i$, for all $i \in [n]$. The model $\Upsilon_\Phi$ is rather canonical, but is not (in general) equal to the BSZ model. However, there is an intimate relationship between these two models that we can give, in terms of the two homomorphisms defined: deconstructing the model $\Upsilon_\Phi$ and then reconstructing the resulting models yields the BSZ model.

**Theorem 6.** *For all true quantified Horn formulas $\Phi$, it holds that the BSZ model of $\Phi$ is equal to* $\mathbf{r}(\mathbf{d}(\Upsilon_\Phi))$.

Theorem 6 gives a purely algebraic characterization of the BSZ model which, as we have seen, was originally defined proof-theoretically.

## References


1. Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas. Inf. Process. Lett. 8(3): 121-123 (1979).
2. F. Börner, A. Bulatov, A. Krokhin, and P. Jeavons. Quantified Constraints: Algorithms and Complexity. Computer Science Logic 2003.
3. A. Bulatov. Combinatorial problems raised from 2-semilattices. Manuscript.
4. Andrei A. Bulatov. A Dichotomy Theorem for Constraints on a Three-Element Set. FOCS 2002.
5. A. Bulatov. Malt'sev constraints are tractable. Technical report PRG-RR-02-05, Oxford University, 2002.
6. Andrei A. Bulatov. Tractable conservative Constraint Satisfaction Problems. LICS 2003.
7. Andrei A. Bulatov, Andrei A. Krokhin, and Peter Jeavons. Constraint Satisfaction Problems and Finite Algebras. ICALP 2000.
8. A. Bulatov and P. Jeavons. An Algebraic Approach to Multi-sorted Constraints Proceedings of 9th International Conference on Principles and Practice of Constraint Programming, 2003.
9. A. Bulatov, and P. Jeavons. Algebraic structures in combinatorial problems. Technical report MATH-AL-4-2001, Technische Universitat Dresden, 2001.
10. Hans Kleine Büning, Marek Karpinski, and Andreas Flögel. Resolution for Quantified Boolean Formulas. Information and Computation 117(1): 12-18 (1995).
11. Hans Kleine Büning, K. Subramani, Xishun Zhao. On Boolean Models for Quantified Boolean Horn Formulas. SAT 2003.
12. A. Bulatov, and P. Jeavons. Tractable constraints closed under a binary operation. Technical report PRG-TR-12-00, Oxford University, 2000.
13. Hubie Chen and Víctor Dalmau. (Smart) Look-Ahead Arc Consistency and the Pursuit of CSP Tractability. Manuscript.
14. Nadia Creignou, Sanjeev Khanna, and Madhu Sudan. Complexity Classifications of Boolean Constraint Satisfaction Problems. SIAM Monographs on Discrete Mathematics and Applications 7, 2001.
15. Víctor Dalmau and Justin Pearson. Set Functions and Width 1. Constraint Programming '99.
16. Alvaro del Val. On 2SAT and Renamable Horn. In AAAI'00, Proceedings of the Seventeenth (U.S.) National Conference on Artificial Intelligence, 279-284. Austin, Texas, 2000.
17. Ian Gent and Andrew Rowley. Solving 2-CNF Quantified Boolean Formulae using Variable Assignment and Propagation. APES Research Group Report APES-46-2002. 2002.
18. Peter Jeavons. On the Algebraic Structure of Combinatorial Problems. Theor. Comput. Sci. 200(1-2): 185-204, 1998.
19. P.G.Jeavons, D.A.Cohen and M.Cooper. Constraints, Consistency and Closure. Artificial Intelligence, 1998, 101(1-2), pages 251-265.
20. Peter Jeavons, David A. Cohen, and Marc Gyssens. Closure properties of constraints. J. ACM 44(4): 527-548 (1997).
21. Marek Karpinski, Hans Kleine Büning, and Peter H. Schmitt. On the Computational Complexity of Quantified Horn Clauses. CSL 1987: 129-137.
22. T. Schaefer. The complexity of satisfiability problems. Proceedings of the 10th Annual Symposium on Theory of Computing, ACM, 1978.